

# Teaching alternative mathematics.

## 2: Calculus without Limits

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### Extended Abstract

In my book *Cultural Foundations of Mathematics: the Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16<sup>th</sup> c. CE* (Pearson Longman, 2007), I established that calculus developed in India in the 5<sup>th</sup> c. CE, in relation to agriculture (calendar, astronomy) and navigation,<sup>1</sup> and was transmitted to Europe in the 16<sup>th</sup> c. by Cochin-based Jesuits.<sup>2</sup> This was in connection with the European navigational problem, the foremost scientific challenge before Europe, the solution of which required precise trigonometric values (then available to ten decimal place precision in India).<sup>3</sup>

A noteworthy point: Europeans then were mathematically challenged, since even elementary fractions were absent in Greek and Roman arithmetic. Though Indian arithmetic was imported by Gerbert (10<sup>th</sup> c.) from Arabs in Cordoba, and then by Fibonacci from African traders, it remained with a few Florentine merchants and was introduced in the Jesuit syllabus ca. 1575 by Clavius only after being imported for the third time direct from Cochin. Consequently, Europeans fumbled in understanding Indian calculus and especially did not understand how to sum its infinite series.<sup>4</sup>

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1 “Cultural Foundations of Mathematics” (book review), *Ghadar Jari Hai*, 2(1) (2007), pp. 26–29. <http://ckraju.net/papers/GJH-book-review.pdf>.

2 For an early account, see, C. K. Raju, “Computers, mathematics education, and the alternative epistemology of the calculus in the Yuktibhasa”, *Philosophy East and West*, 51:3 (2001) pp. 325–362. <http://ckraju.net/papers/Hawaii.pdf> (presented Jan 2000). Also, C. K. Raju, “The Infinitesimal Calculus: how and why it was imported into Europe”, abstract of talk at National Institute of Advanced Study, Bangalore, Dec 2000. <http://ckraju.net/IndianCalculus/Bangalore.pdf>. For more recent accounts, see, *Encyclopedia of Non-Western Science, Technology and Medicine*, Springer, 2016, article on Calculus, pp. 1010–1015. <http://ckraju.net/papers/Springer/ckr-Springer-encyclopedia-calculus-1-final.pdf>, and article on “Calculus transmission”, pp. 1016–1022. <http://ckraju.net/papers/Springer/ckr-Springer-encyclopedia-calculus-2-final.pdf>

3 C. K. Raju, *Cultural Foundations of Mathematics: the nature of mathematical proof and the transmission of the calculus from India to Europe in the 16<sup>th</sup> c. CE*, Pearson Longman, 2007.

4 For a quick account and images, see the video C. K. Raju, “Calculus: the real story”, talk at Massachusetts Institute of Technology, Cambridge, Mass, 26 April 2015, <https://www.youtube.com/laodCGDjqzs>. Abstract at <http://ckraju.net/papers/Calculus-story-abstract.html>,

On European superstitions, mathematics is exact, but the exact sum of an infinite series is *physically* impossible. Hence, Descartes<sup>5</sup> declared infinite series (ratio of curved and straight lines, or infinite series for pi) as not mathematics, imagining that all sums could only be carried out physically. Newton thought his conceptually confused metaphysical fluxions<sup>6</sup> (now abandoned) provided the way to exactitude. Later it was the metaphysics of “real” numbers, to construct which a still bigger metaphysics of set theory was required (first Cantor’s set theory, then axiomatic set theory of the 20<sup>th</sup> c.). Colonial education returned calculus with all that load of metaphysics (of nil practical value) wrapping it with a false history (which declared it the work of Newton and Leibniz), and declaring it to be superior, and teaching us to imitate it. Accordingly, the NCERT school math text teach about real numbers (without defining what a real number is) they teach about limits without defining what a limit is and so on. They still stick to the colonial vision that colonial education is about teaching blind imitation of the West to students.

At no stage did the NCERT ever compare the inferior calculus that Europe returned through colonial education with calculus as it originally developed in India, to decide which is really better. We have done so.<sup>7</sup>

Indian calculus began with Aryabhata who invented the numerical method today wrongly called “Euler’s” method to numerically solve a differential equation for the sine function.<sup>8</sup> Slight improvements of this method are still at the heart of the calculus as used in most applications, such as sending a rocket to the moon.

Further, to sum infinite series Indians used what is today called non-Archimedean arithmetic (Brahmagupta’s *avyakt ganit*).<sup>9</sup> With non-Archimedean arithmetic,<sup>10</sup> unlike formal reals, there are no limits, though there are infinities and infinitesimals (not to be confused with imperceptible).

These are to be used with a philosophy of *inexactitude* called zeroism<sup>11</sup> (*sunyavada*): that is, infinitesimals have to be discarded analogous to the way small numbers are discarded in a floating point computation.

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5 R. Descartes, *The Geometry*, trans. David Eugene and Marcia L. Latham, Encyclopaedia Britannica, Chicago, 1996, Book 2, p. 544.

6 For a detailed account of Newton’s fluxions, see *Cultural Foundations of Mathematics*, cited above. They are conceptually confused, because he assumed (in the Scholium to his *Principia*) that “absolute, true, and mathematical time...flows on without regard to anything external”. Believing that time itself flows is already paradoxical as noted by Sriharsa a thousand years before McTaggart, but making time metaphysical was the cause of the failure of Newtonian physics. See, C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, 1994, or an easy account in C. K. Raju, “Time: what is it that it can be measured?”, “Time: What is it That it can be Measured?”, invited plenary talk at the International Pendulum Program Conference, Univ. of New South Wales, Sydney, 13 Oct 2005, *Science & Education* 15(6) (2006) pp. 537–551. Correcting Newton’s error corrects his theory of gravitation. See, C. K. Raju, “Retarded gravitation theory” in: Waldyr Rodrigues Jr, Richard Kerner, Gentil O. Pires, and Carlos Pinheiro (ed.), *Sixth International School on Field Theory and Gravitation*, American Institute of Physics, New York, 2012, pp. 260–276. [http://ckraju.net/papers/retarded\\_gravitation\\_theory-rio.pdf](http://ckraju.net/papers/retarded_gravitation_theory-rio.pdf).

7 C. K. Raju, “Teaching Mathematics with a Different Philosophy. 1: Formal mathematics as biased metaphysics”. *Science and Culture* 77 (2011) pp. 275–80. . arxiv:1312.2099. “Teaching Mathematics with a Different Philosophy. 2: Calculus without limits”. *Science and Culture*, 77 (2011) pp. 281–86. . arxiv:1312.2100.

8 *Cultural Foundations of Mathematics*, cited above, chp. 3.

9 For a quick account, see articles on calculus in the Springer Encyclopedia, cited above.

10 For a quick introduction to non-Archimedean arithmetic, see E. W. Moise, *Elementary Geometry from an Advanced Standpoint*, Academic Press, New York, 1963.

11 See, *Encyclopedia of Non-Western Science, Technology and Medicine*, Springer, 2016, article on Zeroism, pp. 4604–4610. <http://ckraju.net/papers/Springer/zeroism-springer-f.pdf>.

With these three principles (powerful numerical methods vs symbolic elementary functions, non-Archimedean arithmetic vs formal reals, and zeroism vs exactitude) we have successfully taught calculus without limits to ten groups of students in six universities in three countries.<sup>12</sup> The pre-test uniformly shows that the students learn nothing of calculus in schools, but the post-test showed they become experts after the course. I will report on these teaching experiments, and their relation to the recent attempts to decolonise mathematics teaching.<sup>13</sup>

We now propose to integrate it backward with school education since much of the decolonised calculus course can be done at the school level instead of the undergraduate level, and the course integrates nicely with the *rajju ganit* course (to be taught at the level of class IX). The key difficulty here is a political one: that schools, unlike the university, lack any autonomy as regards the syllabus, and those who make the syllabus, have only some colonially-recognized degrees, but lack in-depth knowledge of mathematics and its real-life applications, and stubbornly stick to blind imitation of the West.

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- 12 E.g. CUTS. Sarnath (<http://ckraju.net/blog/?p=34>), CISSC, Tehran (<http://ckraju.net/blog/?p=84>), Universiti Sains Malaysia (four groups, see articles on teaching math with an alternative philosophy above), AlBukhari University, Malaysia, Ambedkar University, Delhi (<http://ckraju.net/blog/?p=83>), SGT University, Delhi NCR (<https://www.facebook.com/media/set/?set=a.835427293287604.1073743087.151010081729332&type=3>).
- 13 E.g., C. K. Raju, “Decolonising math and science education”. In: *Decolonising the University*, ed. Claude Alvares and Shad Faruqi, Universiti Sains Malaysia and Citizens International, 2012, pp. 162–195. <http://ckraju.net/papers/decolonisation-paper.pdf>. “Decolonising math and science education”, paper for International conference on “Modern Transformation and the Challenges of Inequalities in Education in India”, Delhi University 27-29 Nov, 2014. Advance version in “Decolonising math and science education”. *Ghadar Jari Hai* 8(3), 2014, pp. 5–12. [http://www.ghadar.in/gjh\\_html/?q=content/decolonising-math-and-science-education](http://www.ghadar.in/gjh_html/?q=content/decolonising-math-and-science-education). “To decolonise math stand up to its false history and bad philosophy”, censored article, republished most recently in *Rhodes Must Fall*, Zed Books and University of Chicago Press, 2018. “Decolonising math education”, keynote address, 11<sup>th</sup> Higher Education Conference, University Kwazulu Natal, Durban (to appear), video at <https://www.youtube.com/watch?v=cpS6MfzJr2E>, presentation at <http://ckraju.net/papers/presentations/ckr-Durban-keynote.pdf>. “Decolonising math: why it makes science better (and enables students to solve harder problems),” abstract of censored keynote addresses at the 6<sup>th</sup> Palestinian Conference on Modern Trends in Mathematics and Physics, PCMTMP-VI, Palestine Technical University, 5-8 Aug 2018, <http://ckraju.net/papers/palestine-extended-summary.pdf>.