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Revised Version

How should 'Euclidean' geometry be taught?

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Introduction

I grew up on the classical presentation of the *Elements*, as found in books like those by Todhunter. It had one or two confusing points, but the work as a whole had a certain persuasive charm and seductive beauty to it that still lingers with me. The fascinating drawings of Japanese temple geometry[1] are, to my mind, the best example of something that today still evokes that sense of beauty. Some history is needed to understand how geometry has reached the present state of ugliness and confusion in the NCERT texts (especially the text for Class 9), and what should be done to correct it. Tracing the history also helps to arrive at a clearer understanding of the *Elements*, needed for any corrective process.

The Arabic-Islamic tradition of geometry

First, 'Euclidean' geometry is also a strong Arabic-Islamic tradition. There are at least three dozen known commentaries and translations of the *Elements* in Arabic, including those of al Kindî, Thabit Ibn Qurra, al Farâbi, al Haitham, Ibn Sînâ, and Nasiruddin at Tûsî. Unlike 19th century European historians, Arabs did not feel any need to hide the fact that they got their initial knowledge of geometry from others: the *Haji Khalfa* records[2] that Caliph al-Mansur (754–775) sent a mission to the Byzantine emperor, from

[1] *Japanese Temple Geometry Problems*, San Gaku, Selected and translated by J. Fukugawa and D. Pedoe, Winnipeg, Canada, 1989. I am indeed grateful to Prof. E. C. G. Sudarshan for presenting me with a copy of this book.

[2] T. L. Heath, *The Thirteen Books of Euclid's Elements*, Dover Publications, New York, [1908] 1956, Vol. I, p 75. It should be pointed out that Heath has a curiously ambivalent attitude: while he prominently quotes

whom he obtained a copy of the *Elements* among other Greek books. Caliph al-Mamun (813–833) similarly obtained a copy of the *Elements* from Byzantium. According to the *Fihrist*, a late Arabic index of books, al Hajjâj translated it twice in a 'Haruni' (for Harun ar-Rashid 786–809) and a later 'Mamuni' version.

It is necessary here to point out that 'translation' usually meant 'rewriting' the book. This was particularly the case with the *Elements*, because the book apparently never was entirely as elementary as it first seems. As recorded in the *Fihrist*, from the earliest times of Heron of Alexandria, all commentaries and translations of the *Elements* endeavoured "to solve its difficulties", and explain "the obscurities in Euclid".[3] Not the least of these obscurities concerns the name 'Euclid'. The Arabs spoke of 'Uclides', which they derived from *Ucli*, a key, and *des*, measure or particularly measure of earth (= geometry), so that *Uclides* meant *key to geometry*.

Why was Uclides so important to Arabs? Why did so many key Arabic thinkers rewrite Uclides? Why was Uclides a standard part of the curriculum of later Islamic thinkers? The 'standard' histories of geometry, being concerned almost exclusively with the West, do not ever seem to have bothered to raise this question. The importance of the question, in understanding the *Elements*, will become clear later on, when we answer it.

Euclid the geometer: a name or a person?

While the Arabic-Islamic tradition of the *Elements* is quite clear, it is not so clear that there was any actual person called Euclid who wrote the *Elements*. The only Euclid known to classical Greek tradition was Euclid of Megara, a contemporary of Plato. When medieval Europe first came to know about the *Elements* and Aristotle from the Arabs, Europeans thought that Uclides was a reference to Euclid of Megara. This baseless belief about this standard text was taught in Universities like Paris, Oxford, Cambridge for over five *centuries:* the first English translation of 1570, for instance, attributed the *Elements* to Euclid of Megara.[4] The scholarship of the late nineteenth century has

the *Haji Khalfa* at p 75, to establish that Arabs had translated copies of the *Elements*, on p 4 he asserts regarding the "apparently circumstantial accounts of Euclid given by Arabian authors" that "the origin of their stories can be explained as the result of (1) the Arabian tendency to romance, and (2) …misunderstanding." He goes on to assert (p 4) that these accounts were intended "to gratify a desire which the Arabians always showed to connect famous Greeks in some way or the other with the East" and cites (p 4, note 6) the *Haji Khalfa* to conclude that "The same predilection made the Arabs describe Pythagoras as a pupil of the wise Salomo, Hipparchus as an exponent of Chaldean philosophy or as the Chaldean, Archimedes as an Egyptian etc." In short, Heath's attitude is to accept as true, from Arabic sources, whatever suits him, and to reject everything else with some racist remarks.

[3] Heath, p 21.

[4] Heath, p 109.

veered around to the view that it was impossible that Euclid of Megara could have been the author. The reasons for this shift need to be made quite explicit.

If one discounts later Arab sources, as Heath does, our belief in the historicity of Euclid rests wholly and solely on a single remark attributed to Proclus. In this remark, Proclus is not particularly definite about Euclid, for his language admittedly shows that he is the first to speak of Euclid, and is proceeding on speculative inferences about events some seven centuries before his time:

> All those who have written histories [of geometry] bring to this point their account of the development of this science. Not long after these men [Hermotimus of Colophon and Philippus of Mende] came Euclid, who brought together the *Elements*, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. He must have been born in the time of the first Ptolemy, for Archimedes [who comes after the first Ptolemy] mentions the *Elements*; and further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says.[5]

If Proclus is right and Euclid was much younger than the pupils of Plato, then he could not possibly have been Euclid of Megara, a contemporary of Plato. If, however, Proclus is wrong about the date of Euclid, we could well conclude that he was also confused about the person, in this vague paragraph, so we would be left with no basis to believe in any person called Euclid. (The story about there being no royal road to geometry has been told also about Alexander and Menaechmus; the relation of political equality to the geometric equality in the *Elements* is considered later.)

Prior to Proclus, this Euclid, if at all there was such a person, did not have the stature that he acquired in later times through the combined influence of Islamic and Christian rational theology, and colonial history. *No author prior to Proclus mentions Euclid*, though there are references to other historians of geometry like Eudemus, Eudoxus, and Apollonius, a carpenter of Alexandria, who, according to Arab sources, is said to have written a book in 15 sections to make geometry accessible to all. Claudius Ptolemy, for example, does need to use geometry in the *Almagest*, e.g. the theorem of Menelaus, but he makes no mention of Euclid, even though the Great Library of Alexandria was still intact in Ptolemy's time, and there is ample evidence that he not only consulted it but relied on it heavily for his astronomical 'observations'. It is unconvincing to assert that Ptolemy had no need of the *Elements* since they were, in a sense, elementary. All known commentaries on the *Elements*, such as those of Heron, Porphyry, and Pappus, directly

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[5] Proclus, *A Commentary on the First Book of Euclid's Elements*, (Tr) Glenn R. Morrow, Princeton University Press, 1992, p 56. Heath p 1, and footnotes 2 and 3. Heath omits the first sentence. His footnote 2 asserts that the word γε'γονε "must mean" "flourished" and not "was born", on the grounds that "otherwise part of Proclus' argument [for the existence of Euclid] would lose its cogency".

or indirectly mentioned in the Arabic literature, postdate Claudius Ptolemy who comes over two centuries after Cleopatra, the last of the Ptolemies who ruled Egypt.

In his commentary on Ptolemy, Theon of Alexandria (c. 4th century CE), too, does not mention Euclid. In the same context, Theon, however, does refer to *his* book on the *Elements*:

> that sectors in equal circles are to one another as the angles on which they stand *has been proved by me in* my *edition of the Elements at the end of the sixth book*. [6]

Proclus himself acknowledges, (in the beginning of the quotation) that he is the first person to mention Euclid, stating that Euclid is NOT mentioned by earlier historians of geometry. So, is this quote from Proclus adequate to establish the historicity of Euclid or the antiquity of the *Elements?* Imagine for a minute that we are dealing with Arab tradition rather than Greek tradition, and apply to Greek tradition, the standards of critical historiography that Heath applies to Arabs. What would be the conclusion? If one is not a rank racist, the least one can do is to explore alternatives to the traditional belief in the historicity of Euclid and the antiquity of the *Elements*. Perhaps Proclus simply misjudged the antiquity of the *Elements*, like later Arabs misjudged the antiquity of Proclus' works.

It is also possible that Proclus attributes authorship to Euclid in the same way as later Arabic texts attributed various works, including the works of Proclus, to Aristotle—after all attributions were not so terribly important either to the Neoplatonists or to the Islamic rational theologians, as they were to later-day European historians of science, or as they are to current-day information capitalism, where ownership is decided on attribution. Arabic treatises customarily began by taking the name of Allah, and after that attributing everything to a famous early source. This custom can still be observed in relatively remote places like the Lakshadweep islands where it has survived. The custom of attributing everything to an early source—the earlier the better—was a form of homage, and added authority to the text; it was not meant to be taken literally. Among Greeks, Pythagoreans followed this custom of attributing everything to Pythagoras, and the continuity of Pythagoreans with Neoplatonism is well known.

Mathematics and Religion

The most plausible alternative, however, is the following. Given the politics of the Roman empire in his time—with violent priest-led Roman-Christian mobs attacking Neoplatonists, murdering the most brilliant among them like Hypatia, and invoking state-support to smash or takeover Neoplatonic places of worship,[7] and burn down the

[6] Heath, p 46; emphasis Heath's, de-emphasis mine.

[7] e.g. in 390 the temple of Seraphis and the adjacent library of Alexandria were burnt down by a violent

Great Library of Alexandria—it would have been quite natural for Proclus, or someone else between Claudius Ptolemy and Proclus, to have simply invented a Greek called Euclid to give an appropriate pedigree to their own teaching. In this context one should recognize that mathematics was then viewed not as a 'universal' or 'secular' science, but as a key aspect of the religious and political philosophy of Neoplatonism. The chief aim of Proclus' prologue to the *Elements* is to bring out this dimension of mathematics which he felt was neglected by some of his contemporaries.

> Pythagoreans recognized that everything we call learning is remembering,…although evidence of such learning can come from many areas, it is especially from mathematics that they come, as Plato also remarks. "If you take a person to a diagram," he says[Phaedo 73b], "then you can show most clearly that learning is recollection." That is why Socrates in the Meno uses this kind of argument. This part of the soul has its essence in mathematical ideas, and it has a prior knowledge of them….[8]

(The famous Socratic argument was as follows.

> The soul, then, as being immortal, and having been born again many times and having seen all the things that exist, whether in this world or in the world below, has knowledge of them all; and it is no wonder that she should be able to call to remembrance all that she ever knew about virtue and about everything; for as all nature is akin, and the soul has all things, there is no difficulty in her in eliciting or as men say learning out a single recollection all the rest, if a man is strenuous and does not faint; for all enquiry and all learning is but recollection. [9]

Socrates then gave a practical demonstration of this by questioning a slave boy and eliciting the right responses regarding geometry.)

For Proclus, then, mathematics was not a 'secular' activity, but the key means of propagating his fundamental religious beliefs. This is the concluding thought of part I of his prologue:

> This, then, is what learning (μα′θεσιζ [mathesiz]) is, recollection of the eternal ideas in the soul; and this is why the study that especially brings us the recollection of these ideas is called the science concerned with learning (μαθεματικη′[mathematike]). Its name thus makes clear what sort of function this science performs. It arouses our innate knowledge…takes away the forgetfulness and ignorance [of our former existence] that we have from birth,…fills everything with divine reason, moves our souls towards Nous,…and through the discovery of pure Nous leads us to the blessed life.[10]

Christian mob. The magnificent temple of Dea Caelestis at Carthage remained open until c. 400; but many laws were passed against pagan temples, and, in 401, the synod of Carthage twice asked the State to implement these laws. Eventually, in 407 the Catholics forcibly took possession of Dea Caelestis and Bishop Auerilus, Augustine's lifelong friend, triumphantly planted his cathedra at the exact spot occupied by the statue of the pagan goddess. H. Jedin and J. Dolan (eds) *History of the Church*, Vol. II *The Imperial Church from Constantine to the Early Middle Ages*, Tr. Anselm Biggs, Burns and Oates, London, 1980, p 205.

[8] Proclus, cited earlier, 45, p 37.

[9] Plato, Meno, 81–83.

[10] Proclus, cited earlier, 47, p 38.

These religious beliefs were earlier championed within the Christian church by Origen (also of Alexandria, and from the same school as Proclus). However, by Proclus' time, these religious beliefs ('doctrine of pre-existence', equity) were exactly what were being abusively opposed and cursed by the church and its key ideologues (Augustine, Jerome, Justinian). It is well known that fundamental aspects of present-day Christian religious dogma, such as resurrection (as opposed to 'pre-existence'), *eternal* (as opposed to temporary) heaven and hell, doctrine of sin (as opposed to essential equity), etc., came about from the rejection of Origen and the acceptance of Augustine during this period, starting from Constantine and ending with Justinian.

Therefore, Proclus, in writing on mathematics from the philosophical viewpoint, was right at the eye of a religious storm, at its dead centre in Alexandria, and exposed to great personal risk. Since Jerome had only just translated the Bible from Greek into Latin, and Greek was still held in high regard in the Roman empire, inventing the name Euclid, to give an early Greek legacy to his teachings would have been the most natural strategy for Proclus.

If 'Euclid' was indeed invented to escape from religious persecution, then it would, have been entirely in keeping with the character of the Egypto-Greek Mysteries, if the name 'Euclid' had some 'mysterious' significance, as the Arabs thought. Proclus' fears, incidentally, were quite genuine, for soon after him, the school at Alexandria was permanently shut down, at about the time that Justinian cursed Origen of Alexandria.

Can authorship be attributed to a single individual?

There is another way of looking at the question of authorship. It is clear that, from at least the time of Theon and Proclus, through the Arabic and European rationalists, right down to the time of Hilbert, Birkhoff, and the US School Mathematics Study Group, there has been a continuous attempt to remove the obscurities in the *Elements*, and to update it. To look for a unique author for the *Elements* is like trying to trace the origin of all the water in a mighty river back to its visually apparent source in a small pond: this transparently neglects the vast underground drainage system that contributes most of the water to the river on its way to the sea.

As for the apparent source itself, Europe got its knowledge of the *Elements* from the decaying Arab empire, the Arabs got their knowledge of the *Elements* from the decaying Roman empire, the Romans got their knowledge and culture from the decaying Greek empire, and the Greeks, as Herodotus records, got their knowledge of geometry from the Egyptians. As I have argued in detail, elsewhere,[11] the typical pattern is that the direction in which information flows has been *from* the vanquished *to* the military victor,

[11] C. K. Raju, 'Interaction between India, China, and Central and West Asia in Mathematics and

though this fact has often enraged the descendants of the military victors. There is ample evidence that 18th–20th century CE European historians of science reinvented history in a racist[12] way to make it appear that this entire chain of information transmission had a unique beginning in Greece. These historians did not represent the Greek texts as merely one in a chain of translations and improvements into English, from Latin, from Arabic, from Greek, and from Egyptian texts, but represented the Greek text as the absolute beginning of this chain—as the original creative fount of practically all human thought! Since the geographical origin of the *Elements* (and all its earliest commentaries) in Alexandria, in the African continent, could hardly be denied, the name Euclid, suggesting a Greek legacy, was critical to the process of appropriation via Hellenisation.[13]

Why was this appropriation first attempted? Why were the *Elements* so important to the rational theologians of Christianity? This is a complex issue to which we will return when we address the importance of the *Elements* for Islamic rational theology.

The most recent clarification of obscurities in the *Elements*

Let us first examine the *most recent* example of clarifying obscurities in the *Elements*. In recent times, a major step to modify the teaching of 'Euclidean' geometry was taken in 1957 when the US School Mathematics Study Group issued its recommendations on the teaching of geometry.[14] That recommendation, followed the studies into the foundations of geometry by Hilbert,[15] Russell,[16] and Birkhoff,[17] etc. These authors addressed a variety of obscurities in the *Elements*. The most obvious of these obscurities

Astronomy' (to appear) in A. Rahman (ed), PHISPC, New Delhi, 1999.

[12] Martin Bernal, *Black Athena: The Afroasiatic Roots of Classical Civilization*, Vol. 1: *The Fabrication of Ancient Greece 1785-1985*, Vintage, 1991. The use of the term racist, as distinct from Spengler's term 'Eurocentric,' refers *also* to the technology gap and the industrial revolution. See, M. Adas, *Machines as the Measure of Men*: *Science, Technology and Ideologies of Western Dominance*, Oxford, New Delhi, 1990. While Bernal does not say much about the history of science *per se* (and neither do his detractors in the more recent debate in *Isis*), it is clear that the resurrection of Euclid, after the belated discovery that he could not have been Euclid of Megara, is very much in line with the belief in a nineteenth century pattern of fabricating a Greek origin for everything under the sun. A closer look at the material basis (palimpsets etc.) of the conclusions of classical scholars will make clear the enormous amount of tinted speculation that underlies this belief.

[13] The Hellenisation itself proceeded by reference to the military conquests of Alexander and Julius Caesar, and the in-between period of Ptolemaic rule. Consequently the importance of these conquests got amplified out of all proportion to their global or even local significance.

[14] *School Mathematics Study Group: Geometry*, Yale University Press, 1961.

[15] D. Hilbert, *The Foundations of Geometry*, Open Court, La Salle, 1902.

[16] B. Russell, *The Foundations of Geometry*, London, 1908.

[17] G. D. Birkhoff, 'A Set of Postulates for Plane Geometry (based on scale and protractor),' *Ann. Math.* **33** (1932).

may be put into the following classes.
(1) **Unsound definitions:** e.g., those of point, line, plane etc.
(2) **Missing definitions:** but the corresponding notions are used: e.g. area.
(3) **Hidden assumptions:** e.g. the correspondence of lines with real numbers.
In addition to these, there are subtler problems, relative to the current formalistic notion of mathematics, such as
(4) **Axioms taken as self-evident truths** (about empirical reality): this is also true of the constructions used in proofs.
(5) **Redundant assumptions:** e.g. the parallel postulate becomes redundant if one admits reals and rigid motions, or the notion of distance.

In judging these obscurities in the light of current formalistic mathematics, one must, of course, keep in mind that the present-day formalistic epistemology of mathematics (axiom-definition-theorem-proof) itself historically originated from the analysis and clarification of these obscurities in the *Elements*. Furthermore, one must also bear in mind that there is nothing universal or 'natural' about the formalistic approach, and that it is steeped in a particular theological and cultural tradition.[18]

The unreal and meaningless as the sole concern of mathematics

The obscurities of type (1) are clear enough. One can define something ostensively (e.g. one can define the word 'dog' by pointing to an instance of a dog) or one can define it in other words. In the case of a geometric point, an ostensive definition seems somewhat unsuitable: Platonic philosophy requires that geometry should deal with idealisations that have no real existence. Hence one cannot point to a point. One *can* point to a dot on a piece of paper; but no real entity like a dot can ever correspond to the ideal notion of a geometric point which is required not to have any real existence.

The alternative is a verbal definition. Consider the definition in the *Elements:* "A point is that which has no part, or which has no magnitude." (The 'Heiberg' version has only the first part of this definition.) A person familiar with atoms and magnitudes may not question this definition: but it communicates nothing to anyone else. (Besides, is one talking of *real* atoms here—elementary particles of some sort? The particle which is closest to a point is the electron. But the electron cannot be a Euclidean point, for a circuit around a Euclidean point brings us back to where we started, whereas *two* circuits around the electron are needed to return to the starting point, because the electron has the paradoxical property of half-integral spin.) Clearly, a verbal definition of a non-real

[18] C. K. Raju, 'Mathematics and Culture,' in: Daya Krishna and K. Satchidananda Murty (eds) *History, Time and Truth: Essays in Honour of D. P. Chattopadhyaya*, Kalki Prakash, New Delhi 1998. Reprinted in *Philosophy of Mathematics Education* **11** (1999). Available at http://www.ex.ac.uk/~PErnest/

notion cannot avoid an infinite regress, for at no point can it terminate in an ostensive definition.

Thus, Platonic philosophy, by its insistence on the non-reality of the ideal, eliminates both possibilities of an ostensive or a verbal definition, and the only option left is that of current formalistic mathematics, which regards the notions of point, line, etc. as meaningless, undefined notions. In other words, the current way of removing the obscurities in the *Elements* is to adopt Russell's definition of mathematics: "Mathematics may be defined as a subject in which we never know what we are talking about…"![19]

Real numbers and Euclidean proportions

Obscurities of type (2) are examined later. Obscurities of type (3) are manifest in the very first proposition of the *Elements*. The first proposition constructs an equilateral triangle on a given segment AB. This process involves drawing two circles, the first with centre at A and radius AB, the second with centre at B and radius BA. One obscurity is that the two circles may fail to intersect, in the sense that the point of intersection need not mathematically exist. If points on the circles correspond to (pairs of) rational numbers, there may be 'gaps' between them, such as the gaps between the numbers 1, 2, 3. Indeed one is led to expect such gaps since the 'Euclidean' approach to proportions suggests a reluctance to use irrational numbers like $\sqrt{2}$. It was the attempt to clarify this obscurity in the first proposition of the *Elements* that led Dedekind to the idea of the real line as something that could be 'cut' without leaving any gaps. Needless to say, the real numbers, as conceptualized by Dedekind are something necessarily unreal, for there is no real process by which one can specify or fully name a real number such as π.

The SAS theorem/postulate

The other obscurity in the proof of Proposition I.1 is this: why is the radius measured out *twice*? Can't the first measurement of AB be re-used for BA? This is related to the key obscurity concerning Proposition I.4. This difficulty must have been noticed by every schoolchild who did geometry using the older 'Theonine' texts, like those of Todhunter, current in India up to the end of the 1960's. In the 'Heiberg' version, Proposition 4 of the *Elements* states that

> If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be

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[19] The best that one can do is to *interpret* these meaningless notions using other meaningless notions like sets: e.g. a point is an element of a set, a line is a subset etc.

equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.[20]

In brief: if two sides and the included angle of one triangle are equal to those of another triangle, then the two triangles are equal. We will refer to this as the Side-Angle-Side proposition, or SAS for short.

The key obscurity is this. In the *Elements* the *proof of this proposition involves superposition:* it involves picking up one triangle, moving it through space, rotating it as necessary, and applying it to the other triangle. The later theorems on the equality of triangles (with the exception of I.8) do not, however, use this procedure: they rely instead on SAS.

There is no doubt at all that physical motion in space is implied, and there is a specific Common Notion or Axiom to enable this proof to go through. Common Notion 4 of the 'Heiberg' version asserts "Things which coincide with one another are equal to one another". [21] For those accustomed to reinterpreting this in terms of congruence, it should be pointed out that this clearly applies to distinct geometrical objects that are brought into contact, and superposed, through motion. Likewise, Axiom 8 of the 'Theonine' version asserts: "Magnitudes which coincide with one another, that is, which fill the same space, are equal to one another." If this is not a tautology, it must refer to distinct object which are made to coincide with each other, by moving them about.

Physical movement and motion without deformation

The doubt that must have entered the mind of every schoolchild is the following. This method of picking and carrying greatly simplifies the proofs of all other theorems and riders: if it can be used in one place, why can't it be systematically used in other places as well? My teacher had no satisfactory answer why it was all right to do this in one place, but wrong to do it elsewhere. He simply said it is better not to do it, but could not explain why. But one may attempt an answer as follows.

Picking and carrying line-segments is a common enough thing: one must do this every time one ordinarily makes a measurement. By the late 19th century mathematicians were sceptical about the very possibility of making a measurement: moving an object might deform it. What sense did it make to say that a figure remained identical to itself as it was moved about in space? A shadow moving on uneven ground is continuously deformed; perhaps space itself is similarly 'uneven', so that any motion may involve deformation, and measurement may require more complicated notions like a metric

[20] Heath, p 247.

[21] Heath, p 224 et seq.

tensor. The avoidance of picking and carrying in the proofs of the subsequent theorems was interpreted, by the 20th century, as an implicit expression of this doubt about the very possibility of measurement. It was argued against Helmholtz that measurement required (a) the notion of motion; furthermore this motion must be without deformation, so that it required (b) the notion of a rigid body, and neither of these was the proper concern of the geometer, who ought to be concerned only with motionless space. (The notion of rigid body depends on physical theory; e.g. the Newtonian notion of rigid body has no place in relativity theory, for a rigid body would allow signals to travel at infinite speed.)

Historically, this doubt about measurement was expressed as a doubt about (a) the role of motion in the foundations of mathematics, and (b) the possibility and meaning of motion without deformation. In favour of (a) the authority of Aristotle was invoked to argue that motion concerned astronomy, and that mathematics was "in thought separable from motion". The authority of Kant was implicitly invoked to argue that motion was not *a priori*, but involved the empirical, and *hence* could not be part of mathematics. All these worries are captured in Schopenhauer's criticism of the 'Theonine' Axiom 8 (corresponding to the 'Heiberg' Common Notion 4) which supports SAS:

> …*coincidence* is either mere tautology, or something entirely empirical, which belongs not to pure intuition, but to external sensuous experience. It presupposes in fact the mobility of figures; but that which is movable in space is matter and nothing else. Thus, this appeal to coincidence means leaving pure space, the sole element of geometry, in order to pass over to the material and empirical.[22]

In short, motion, with or without deformation, brought in empirical questions of physics, and Plato, Aristotle, and Kant, all concurred that mathematics *ought* not to be based on physics, but *ought* to be *a priori*, and that geometry *ought* to be concerned only with immovable space.

The synthetic and the metric axiom sets

The Hilbertian reading of the *Elements*, hence denies the possibility of measurement, so that the proof of Proposition 4 (SAS) fails. To preserve the structure of the *Elements* it is then necessary to assume Proposition 4 as a postulate (the SAS postulate) that cannot be proved from any more basic principles. This approach is called the **synthetic approach**.[23] One way to describe this approach is by distinguishing synthetic instruments from those found in the common instrument box of school geometry. The

[22] Schopenhauer, *Die Welt als Wille*, 2nd ed, 1844, p 130, cited in Heath, p 227.

[23] For a detailed and easily accessible account, see E. Moise, *Elementary Geometry from an Advanced Standpoint*, Addison Wesley, Reading, Mass., 1963; B. I. Publications, Bombay, 1966.

synthetic instruments are the straight edge (*unmarked* ruler) and '*collapsible* compass'. The last term is De Morgan's graphic description of the impossibility of measurement with the synthetic approach: distances cannot be reliably picked and carried because the synthetic compasses are loose and 'collapse' as soon as they are lifted from the paper. ('Collapsible compasses' may well be an accurate description of the then-prevailing state of technology!) Hence, also the ruler is left unmarked. In this synthetic approach, the term *equal* used in the 'original' *Elements* is changed to the term *congruence*: motion is replaced by a mapping, so that it is not necessary to transfer figures from one place to another, one only needs to shift one's attention from one figure to the other.

The other way of clarifying the obscurity in the original *Elements* is to accept the possibility of measurement, and to accept that the proof of Proposition 4 (SAS) is valid. This is called the **metric approach**, and has been championed by Birkhoff. The main problem with a full metric approach is that it completely devalues the *Elements*. Even Proclus does not claim any originality for his Euclid; the value of the *Elements* derived from the nice arrangement of the theorems, so that the proof of any theorem used only the preceding theorems. With a full metric approach, even the arrangement of theorems in the *Elements* loses its significance: it is quite possible to prove the 'Pythagorean Theorem' (I.47), by cutting, picking and carrying, without recourse to the preceding theorems.

The synthetic and metric approaches being so different, the problem is to choose one of them.

It is in deference to the synthetic formulation of the *Elements*, that the proposition 4 of the 'original' *Elements* is now taught as the Side-Angle-Side (SAS) *postulate*. This permits one to continue teaching the *Elements* as a valid example of the deductive method of proof used in modern mathematics.

This is unacceptable for several reasons.

(1) A metric approach makes 'Euclidean' geometry very simple: a straightforward metric approach could prove the 'Pythagorean' 'Theorem' (Proposition I.47) in one step, as in the *YuktiBhâsâ* proof.[24] The synthetic approach was originally motivated by the desire to *justify* the apparently needless complexity of the proofs in the 'original' 'Euclid'. The justification was needed because of the importance attached to this text by Christian rational theology. The justification was sought by denying the possibility of picking and carrying segments without deformation; hence, also, the possibility of measurement was denied. Thus, the synthetic approach makes proofs more difficult, and is counter-intuitive—for it denies the everyday ability to pick and carry, and compare and measure. (The ultimate justification for denying the manifest flows from the Platonic-Kantian

[24] K. V. Sarma (Ed and Tr.) *The GanitaYuktiBhâsâ of Jyeshtadeva* (to be published). For a description of the proof, see , C. K. Raju, 'Mathematics and Culture', cited earlier.

idea that mathematics is *a priori*, and so *ought* not to be contaminated by the empirical. The other way of looking at this idea is that it demands that mathematics *ought* not to correspond to anything real, and hence *ought* to remain perfectly meaningless.)

(2) The synthetic interpretation of the *Elements* substitutes the key term *'equal'* in the 'original' by the new term *'congruent'*. This key substitution clearly does *not* work beyond Proposition I.34. Thus, Proposition I.35 states that "Parallelograms on the same base and in the same parallels are equal to one another." This proposition asserts the equality of areas that are quite clearly non-congruent (when not identical). It follows that *one must either abandon all propositions after proposition I.35 (including the 'Pythagorean' 'Theorem' I.47), or else one must abandon the synthetic interpretation* of the *Elements*. It does not help to try to define a general area through triangulation, as Proclus' contemporary, Aryabhata did[25] since the notion of area is not defined anywhere in the *Elements*, and the usual formula for the area of a triangle is itself derived from I.35. Some attempts have been made to supplement the synthetic approach by axiomatically defining area in a way analogous to the Lebesgue measure (overlooking the connection of the Lebesgue measure to the notion of distance). Area, however, is an intrinsically metric notion; indeed, it would be a rather silly enterprise to define area without first defining length.

The schizophrenic method of denying metricity until proposition I.35, and admitting it thereafter is only confusing to young minds. The whole project is born of the compulsions of theology and racist history.[26]

The current text

We have substituted this with our own schizophrenic project. The schizophrenia derives from multiple inheritance. The formal structure of our educational system: schools, colleges, universities is patterned on the system prevalent in Europe, rather than the indigenous tradition of *pathshâlâ*-s or Nalanda and Takshashila. The educational system in Europe was for several centuries quite explicitly oriented towards theological concerns. With the rise of industrial capitalism, in the last hundred years or so, there was a partial shift in the West towards more practical and utilitarian concerns. 'Euclidean' geometry, for example, is no longer taught in British schools.

Independent India accepted industrial capitalism, and the elite in this country still continue to regard education as a means of forging links to the metropolitan centre, so that even 50 years after independence most of the country remains illiterate, and

[25] *Ganita* 6–9. *Aryabhatiya of Aryabhata* (Eds and Trs) K. S. Shukla and K. V. Sarma, INSA, New Delhi, 1976, pp 38–45.

[26] Martin Bernal, *Black Athena*, cited earlier.

education remains the preserve of the elite for one excuse (shortage of government funds) or another (need to commercialise). Education, furthermore, has been 'de-moralised', and the theological concerns of the West have been substituted by elitist chauvinism.

In line with the British legacy of bureaucracy, and the clerk's *dharma* of evading responsibility, our school texts are produced in clerkdom (which still controls education), by a duly constituted committee. The committee has sought to balance the requirements of industrial capitalism (which needs the products of education), with those of chauvinistic history (which seeks to correct racist history without understanding tradition).

These contradictory requirements are reflected in the current NCERT text for Class 9[27]. On the one hand, this is how the NCERT text justifies the teaching of geometry "For instance, those of you who will become engineers, technicians and scientists will not only find all this information useful but will also realise that you cannot do without it." (Needless to say, there is no other concrete instance in the 'explanation' which occupies one paragraph in this vein of redundancy improving communication!) But if practical usefulness were the sole justification for teaching geometry, then metric geometry ought to be taught. Engineers, technicians and scientists, all, have no use for geometry without measurement. (Not even relativists care much for spacetime geometry based on the connection rather than the metric.)

On the other hand, a similar conclusion follows from the historical assertions with which the NCERT exposition of geometry begins [pp 123–124].

> The *Baudhayana Sulbasutras*…contains [sic] a clear statement of the so-called Pythagoras theorem. The proof of this theorem is also implicit in the constructional methods of the *Sulbasutras*.

The subtle way in which Western historians have exploited the notion of 'proof' seems to have quite escaped the authors of the text. Western historians have readily conceded that Babylonians, Egyptians, Chinese, Indians all knew earlier *that* the Pythagorean theorem was true. They have maintained, however, that none of them had a proof, hence none of them knew *why* it was true: they knew of the theorem only as an empirical fact which they did not quite comprehend, much as an ass might know the theorem without comprehending it. Comprehension, therefore, still dawned with the Greeks. To refer to constructional methods as implicit proofs is to miss the central issue clarified above: the motivation for synthetic geometry is that empirical knowledge is not only distinct from mathematics but that it cannot logically precede mathematics. Hence, if the second sentence in the above quote is true, then the very notion of mathematical proof would need to be changed to accept empirical inputs. Needless to say, the committee does not intend any such revolutionary challenge to mathematical authority which is entirely beyond its terms of reference!

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[27] A. M. Vaidya et al, *Mathematics: A Textbook for Secondary Schools, Class IX*, NCERT, 1989, Ninth Reprint Edition (sic) 1998, p 124.

Therefore, on the third hand (surely committees have at least three hands!), the text lapses back into the synthetic geometry recommended by the US School Mathematics Study Group. Like a proper committee report, the resulting text has included a little something to suit every taste. So the text introduces the SAS postulate [p 162] as the "SAS (Side-Angle-Side) Congruence Axiom", where 'axiom' is to be understood as follows [p 125]: "basic facts which are taken for granted (without proofs) are called *axioms*. Axioms are sometimes intuitively evident." That is, an axiom, like a fact, belongs to the domain of empirical and physical, rather than the intuitively *a priori*—exactly the thing that was denied to motivate the SAS postulate and the notion of congruence in the first place! One wonders why, unlike most other committee reports, this report was not left to gather dust!

The natural casualty is the student who has to digest the *whole* thing, and so may be put off geometry for the rest of his life, especially if he is clear-headed. If congruence is explained through superposition ('Heiberg' Common Notion 4, or 'Theonine' Axiom 8), as the text does (p 159–161), one has clearly a metric approach. Within a metric approach, it is trivial to prove the synthetic congruence results proved in the text—in fact there is then no need for a SAS congruence *axiom*, one has a SAS *theorem*, the way it was proved in the 'original' *Elements*. To now prove these results, in the manner of synthetic geometry, on the ground that one is teaching the axiomatic method, is to teach the axiomatic method as a completely mindless and elaborate ritual that one must complete on the strength of the state authority that NCERT enjoys. What children are being taught is not the sceptical attitude which underlies the need for a proof, but mindless obedience to rituals which cannot be justified.

The *khichdi* geometry in the NCERT text for Class 9 is indigestible because it has mixed up the *Elements* by mixing up elements that ought not to be taken together—like diazepam and alcohol—unless the object is to induce a comatose state. To make the text digestible, one needs to sort out *which* geometry one wants to teach: metric, synthetic, or traditional. Even if one wants to teach all three they should be kept in separate compartments: it is NOT a good idea to make the synthetic notion of congruence more intuitive by defining it metrically as the NCERT text does! The authors need to appreciate the incompatibility of the metric and synthetic approaches, and the way these differ from the traditional approach, which incorporates an altogether different notion of mathematical proof.[28]

[28] For the traditional notion of *pramâna* in relation to mathematics, see C. K. Raju, 'Mathematics and Culture', cited earlier.

Traditional geometry distinguished from the metric and the synthetic

Enough has been said above about the incompatibility of the metric and synthetic approaches; and I will briefly recapitulate the way in which both these approaches differ from the traditional approach. First, the authoritative traditional literature is the *sutra* literature; the *sutra* style is well-known for its extreme brevity—like a telegraphic message, further distilled by digital compression. The *sutra*-s are not intended to serve primarily a pedagogical function, and they are not intended to be accessible to all. Consequently, they have no place for proofs. Texts dealing with rationale, on the other hand, being less authoritative, have not been translated. The key text on rationale, available in English translation,[29] is the *YuktiBhâsâ*, which, as stated earlier, proves the 'Pythagorean theorem' in one step, by drawing the figure on a palm leaf, cutting it, and rearranging the cut parts. An examination of rationale in traditional geometry shows the following.

What distinguishes traditional geometry from both metric and synthetic geometry is the traditional notion of proof (*pramâna*). Though there have been many debates in tradition on what constitutes *pramâna*, the one ingredient that went unchallenged was the physically manifest (*pratyaksa*) as a means of proof. The traditional notion is not embarrassed by the empirical, and does not regard it as intrinsically inferior to metaphysics. Both the Baudhayana and the Katyayana *sulbasutra*-s begin by explaining the use of the rope for measuring areas. Aryabhata defined 'horizontal' using a water level, and a 'perpendicular' using a plumb line. The proofs in the *YuktiBhâsâ* clearly accept the physically manifest as a good argument. All this would horrify a modern-day mathematician, who believes that mathematics is *a priori*, and certainly logically prior to the physically manifest.

Asserting the *sulbasutra* tradition would clash with the entire tradition of education in medieval and renaissance Europe, which was geared to theological purposes, and hence reinforced the philosophy of the authorities like Plato, and later Kant which justified the deprecatory attitude towards the physical world. For Proclus, the key object of teaching mathematics was not its military or political utility, which he regarded as subsidiary, but its ability to make the student forget the practical concerns of everyday life and thereby discover his real self.

> the soul has its essence in mathematical ideas, and it has a prior knowledge of them…and brings of them to light when it is set free of the hindrances that arise from sensation. For our sense-perceptions engage the mind with divisible things…and…every divisible thing is an obstacle to our re-

[29] Another text dealing with rationale, the *Karanapaddhati* is now available in a Japanese translation, being retranslated into English.

> turning upon ourselves. … Consequently when we remove these hindrances…we become knowers in actuality.…[30]

Rejecting this attitude is not a trivial matter for all of current-day mathematics depends upon the belief that mathematics is *a priori* and divorced from the empirical.

Nevertheless, the fact is that the Nayyâyikâ notion of proof proceeds from a realistic philosophical standpoint directly opposed to Platonic idealism. Classical Indian tradition saw no need to regard mathematics as something necessarily metaphysical, and consequently, there was no need for two separate procedures of validation: (1) a notion of mathematical proof, and (2) criteria (such as logical and empirical falsifiability) to decide the validity of a physical theory. Therefore, though metric, traditional Indian geometry does not need to proceed from Birkhoff's axioms. Against this background, various other considerations are summarised in Table 1.

The second key point about the notion of proof concerns inference (*anumâna*), about which different schools of thought had mutually different ideas which differed also from the idea of logical deduction underlying the current metamathematical definition of a mathematical proof (which defines a proof as a sequence of statements each of which is either an axiom or is derived from some preceding axioms by the use of *modus ponens* or similar rule of reasoning). The Lokâyata explicitly rejected inference, at least in the metaphysical domain (which includes modern mathematics), allowing its use only for practical purposes. The Buddhist and Jaina traditions pose an even more fundamental question: what should be the logic underlying proof? If one insists on regarding mathematics as metaphysical, as in the current formalistic approach, then what is the justification for the use of a 2-valued truth-functional logic underlying mathematical proof? Clearly, the formalistic approach cannot possibly answer this question—thereby showing that allegedly 'universal mathematical truths' ultimately rest on a narrow base of authority, localised in the West. Despite the authority, the belief is purely a matter of cultural prejudice, for the seven-fold classification (*saptabhanginaya*) of the Jaina *syâdvâda* of Bhadrabahu cannot be accommodated within 2-valued logic, while the 'four-fold negation' used by the Buddha, Nagarjuna, and Dinnaga cannot be accommodated within a truth-functional framework. The logic of the empirical world, by the way, may be similarly quasi truth-functional, for quantum mechanics permits Schrödinger's cat[31] to be simultaneously both alive and dead, without permitting any arbitrary statement to be deduced from this 'contradiction'.

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[30] Proclus, cited earlier, 45.

[31] For details of the relation of quasi truth-functional logic to von Neumann's postulates for quantum mechanics, see C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, 1994.

The objectives of education, and the philosophical substance of the *Elements*

We now have before us, three distinct models of 'Euclidean' geometry: synthetic, metric and traditional. Which model one ought to teach depends upon the objectives of education. The objectives of education in India prior to independence are well known, especially Macauley's objectives of creating a cheap clerical workforce to help rule the empire. In independent India, as things stand, educational objectives have largely been decided in clerkdom by appealing to precedents established in the West. So, before deciding the objectives of education ought to be, it would help to answer the two questions that were postponed earlier. Why were the *Elements* so important to Islamic and to Christian rational theology? Why were they such a necessary part of the theological curriculum? (This is the sort of thing that modern-day mathematicians do not usually understand, since their education, geared to the needs of industrial capitalism, encourages a narrow view of the world, together with an unquestioning acceptance of the postulates and rules of inference laid down by mathematical authority.)

Very briefly, to understand this, one must situate Christian rational theology in the context of the two traditions which it inherited. The first is that of Arabic-Islamic rational theology, which reached medieval Europe through Averröes and the debate that preceded him in Islam, and deeply influenced the beginnings of Christian rational theology.

For the Arab rationalists (Mutâzilâh and *falâsifâ*) Uclides was important as a demonstration of Neoplatonic principles, which they accepted as a key aspect of their theology, attributing it to Aristotle. The Arab rationalists aimed to deduce everything from the two key principles of equity and justice. The *Elements* provided a model of how even the physically manifest could be deduced, starting from the principle of equity. The notion of equality in the *Elements* has obvious political and philosophical overtones of equity, that are quite lost upon those now accustomed to thinking in terms of congruence: the absence of a royal road to geometry was an assertion about the political content of the *Elements*. Equity is contrary to Platonic ideas of the republic, and Proclus' stated aim in writing his commentary on the *Elements* was to inform people about its deep philosophical content—the doctrine of the oneness of humankind.

Secondly, Christian rational theology also inherited the legacy of the early Roman church and its confrontation with Neoplatonism over the issue of equity. Though the very early church doctrines clearly favoured equity, and Origen's theology is barely distinguishable from Neoplatonism, the state-church after Constantine, found this doctrine of equity a gross political inconvenience. We have already noted the church's confrontation with Neoplatonism, ending with the closure of the Alexandrian school and when Origen was formally condemned by the Fifth Ecumenical Council.[32]

Impelled by these contradictory inheritances, Thomist philosophy rejected equity as irrelevant, and retained only the process of rational deduction. The philosophical importance of the *Elements* was now confined to the process of rational deduction which could be used to persuade the non-believer, since both Islamic rationalists and al Ghazâlî accepted that God was bound by Aristotelian logic.

How should geometry be taught in schools?

Against this background, we can finally turn to the question raised in the title of this paper. Which geometry should be taught depends upon the objectives of education. In a democracy these objectives should be decided by consensus, or majority, or, at least, informed public debate. In India, bad governance by the elite has made this impossible. We still follow a tightly hierarchical model: all knowledge is believed to reside at the top management layer, even though it may be manifestly scientifically illiterate, or out of date! Committees are nominated only to make a pretence of oligarchy; but those of us with the slightest experience in committee formation know that the whole structure aims to reflect the will at the very narrow top. This will is that the majority of people in the country should be kept illiterate (so that they do not constitute a threat) and that the cream of educated people should be exported to the West (since that financially benefits the elite from which this creamy layer comes).

Under these circumstances, I cannot prescribe the objectives of education. But once these objectives are laid down, the foregoing should help to arrive at an answer to the question raised in the title of this paper.

(1) If the state policy is that education is justified by its linkages to industrial or information capitalism ("it is needed by future engineers, technicians and scientists") it is not so clear that it is imperative to teach the classical method of proof. We must then consider what is increasingly likely to happen in the future: a computer simulation for which there is no numerical analysis, and no convergence proof. According to Hilbert's ideas this would not count as mathematics. Nevertheless, such computer simulations may be increasingly *used* as the basis of everyday decisions: such as decisions about large financial investments. Briefly, if mathematics is to be justified by its utility, then one should be teaching practical mathematics rather than formal mathematics. In the case of geometry, this means that the synthetic approach should be rejected in favour of the metric approach, and that even with the metric approach, one could omit teaching

[32] or, which amounts to the same thing, thought to have been so condemned for 1400 years. (Hair splitting over the 5th Ecumenical council is irrelevant here.)

proofs. It is true that this might compromise understanding; but if education is justified by its utility, one might as well explicitly accept that understanding is of lesser importance, for the time thus saved could be used to teach some more useful things.

(2) If the objective of education is to establish linkages to tradition, this tradition cannot be arbitrarily selected. The Neoplatonic origin of the *Elements* seems to me undeniable. On the other hand, Neoplatonism links naturally to Indian tradition not only through Islam and the sufi-s, but also through direct contact, and strong conceptual similarities to Advaita Vedanta. The links were physical, with some 250 ships sailing annually to carry out a huge trade with the Roman empire. They were also philosophical: Augustine, born some 50 years after Porphyry's death, records that Porphyry (the very same student of Plotinus, who recorded the *Enneads* and commented on the *Elements*) searched for a universal way for the liberation of the soul in "the *mores* and *disciplina* of the Indi."[33] Therefore, it needs to be spelt out what state policy enables us to say that a certain sort of tradition at a certain point of time should be regarded as more valuable than another tradition at a different point of time. For example, should one reject the Buddhist or Jaina tradition, both of which rejected as wrong many more ancient traditional things? Again, there is no reason why the medieval tradition in which clearly 'Uclides' was part of the *tâlîm*, of, say, Abul Fazl, should not be deemed to be as Indian a tradition as the *sulbasutra*-s. One cannot really say that the more ancient thing is necessarily a more authentic part of one's tradition, for one may quite recently have consciously rejected some ancient ideas like untouchability. Depending upon which tradition is officially approved as worth teaching, one could then decide whether to teach one or more of traditional geometry, or metric geometry (which trivialises the *Elements*), or synthetic geometry and the method of proof (after resolving the issue over the method of proof to be adopted in mathematics).

There is also the following question. Though the geometry of the *sulbasutra*-s has been called 'ritual geometry' because of the association of the *sulbasutra*-s with the construction of *vedis* and *citis*, the fact of the matter is that this geometry had purely practical significance, and lacked the theological orientation of the *Elements*, from the time of Proclus. Practical significance is something that changes from time to time; to teach traditional geometry, devoid of its practical concerns would be to do violence to the tradition by reducing practical considerations to ritualistic one's.

(3) If the objective is to teach a certain method of inference, or a certain method of 'proof', it is not clear that 'Euclidean' geometry is the best vehicle for it. One could take syllogistic examples from elsewhere.

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[33] John J. O'Meara 'Indian Wisdom and Porphyry's Search for a Universal Way' in: R. Baine Harris (ed) *Neoplatonism and Indian Thought*, Sri Satguru publications, Delhi 1982, p 6.

Conclusions

(1) Our current school texts in geometry must be corrected to distinguish clearly between metric and synthetic geometry.

(2) One must decide *which* geometry to teach—metric, synthetic, or traditional—and stick to teaching that geometry. It is NOT a good idea to motivate synthetic concepts like congruence by appealing to the intuitive physical idea of superposition which underlies metric notions.

(3) If traditional geometry is also to be taught, the texts must further separate it from formal metric and synthetic geometry: it is NOT a good idea merely to claim priority, as the present text does, for traditional geometry is fundamentally different, since the traditional notion of proof differs fundamentally from the current metamathematical notion of proof. One should first decide which method of proof one wants to teach, and then develop a mathematics based on that method of proof.

(4) If the aim in teaching the *Elements* is to teach formal axiomatics, the authors of texts should distinguish between meaningless formal axioms and empirical facts. If this is too hard a thing for educators to do, then it is too hard for schoolchildren to understand, and formal axiomatics ought NOT to be taught to schoolchildren.

(5) The *Elements* have long been part of the theological curriculum because of their philosophical significance, first for Neoplatonists (to arouse recollection of one's true Self), then for Islamic rationalists (rational deduction from equity), and finally for Christian rationalists (rational deduction). Our objective in teaching the *Elements* must be formulated in awareness of this significance, as also an awareness of Neoplatonic linkages to Indian traditions directly and via the *sufi*-s.

(6) Our objectives must also recognize that no individual tradition can claim to be the unique Indian tradition either as regards the matter of proof (*pramâna*), or as regards the tradition of geometry: the *sulbasutra*-s, the *YuktiBhâsâ*, and Uclides are all part of Indian tradition. Tradition should not be reduced to ritual by separating it from its original context of practical usefulness.

(7) If we choose to teach geometry purely for its practical utility, then this practical usefulness needs to be clearly thought out in the context of future needs, to protect education from rapid obsolescence.

Table 1: A comparison of metric, synthetic, 'Euclidean' and traditional geometry

Type of geometry	I	II	III	IV
	Metric	Synthetic	'Euclidean'	Traditional
Fundamental setup	(S, L, P, d, m)	(S, L, P, B, ≡)	Semi-idealized (not real, not ideal)	Real space
Distance	d	Not mentioned	Lengths	Measured with a rope
Measure for angles	m	Not mentioned	Only equality and inequality with right angles	Measured physically (e.g. with a plumb line)
Congruence for segments	From d	Given ≡ (for segments)	Not mentioned (only equality, presumed pre-defined)	Not mentioned (equality through 'pick and carry')
Congruence for angles	From m	Given ≡ (for angles)	Not mentioned (only equality, presumed pre-defined)	Not mentioned (only measured equality)
SAS	Theorem	Postulate	Theorem (differently proved)	Similarity and rule of three (equality a special case)
Area	Additional definition needed	Not defined (else length would be defined)	Not defined (only equality, presumed pre-defined)	Explicitly defined through triangulation/rectangulation
Addition Inequality	Real numbers	Congruence classes	Geometric construction	Floating point numbers
Proportion	Real numbers	Congruence classes + complex assertions (using 'betweenness', inequality, and integer addition)	Complex assertions using inequality and integer addition. Not in Book 1	Rule of 3
Instruments	Scale, protractor, and compass ('geometry box')	Unmarked straight-edge and 'collapsible' compasses	Not explicitly stated	Rope

Note: S = set of points, L = set of lines (subsets of points), P = set of planes (subsets of points), d = distance, m = measure for angles, B = 'Betweenness' relation, ≡ = congruence for segments/angles.