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Computers, mathematics education, and the alternative epistemology of the calculus in the *Yuktibhâsâ*

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Abstract

Current formal mathematics, being divorced from the empirical, is entirely a social construct, so that mathematical theorems are no more secure than the cultural belief in 2-valued logic, incorrectly regarded as universal. Computer technology, by enhancing the ability to calculate, has put pressure on this social construct, since proof-oriented formal mathematics is awkward for computation, while computational mathematics is regarded as epistemologically insecure. Historically, a similar epistemological fissure between computational/practical Indian mathematics and formal/spiritual Western mathematics persisted for centuries, during a dialogue of civilizations, when texts on 'algorismus' and 'infinitesimal' calculus were imported into Europe, enhancing the ability to calculate. I argue that this epistemological tension should be resolved by accepting mathematics as empirically-based and fallible, and by revising accordingly the mathematics syllabus outlined by Plato.

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0 Introduction

0.0 The East-West civilizational clash in mathematics: *pramâna* vs proof

In Huntington's terminology of a clash of civilizations, one might analyse the basis of the East-West civilizational clash as follows: the Platonic tradition is central to the West, even if we do not go to the extreme of Whitehead's remark, characterising all Western philosophy as no more than a series of footnotes to Plato. But the same Platonic tradition is completely irrelevant to the East.

In the present context of mathematics, the key issue concerns Plato's dislike of the empirical, so the civilizational clash is captured by the following central question: *can a mathematical proof have an empirical component?*

0.1 The Platonic and Neoplatonic rejection of the empirical

According to university mathematics, as currently taught, the answer to the above question is no. Current-day university mathematics has been enormously influenced by (Hilbert's analysis of) "Euclid's" *Elements*, and Proclus,[2] a Neoplatonist and the first actual source of the *Elements*, argued that

> Mathematics…occupies the middle ground between the partless realities…and divisible things. The unchangeable, stable and incontrovertible character of [mathematical] propositions shows that it [mathematics] is superior to the kinds of things that move about in matter.…Plato assigned different types of knowing to…the…grades of reality. To indivisible realities he assigned intellect, which discerns what is intelligible with simplicity and immediacy, and…is superior to all other forms of knowledge. To divisible things, in the lowest level of nature, that is, to all objects of sense-perception, he assigned opinion, which lays hold of truth obscurely, whereas to inter-

EXTRACT

$$\frac{x \cdot 0 + \frac{x \cdot 0}{2}}{0} = 63$$

This suggests that, when we go beyond the empirical, the 'universal' may lie, as in a physical theory, in what Poincaré called 'convenience'. This criterion of 'convenience' can have profound consequences as in the case of the theory of relativity: the constancy of the speed of light is not an empirical fact (though elementary physics texts usually misrepresent it as such), Poincaré defined the speed of light as a constant as a matter of 'convenience'. I see this criterion of 'convenience' as more modest than the criterion of beauty which seeks to globalize a local sense of aesthetics.

3 History of the calculus

If mathematics is a social construct, which changes with changing social circumstances, then the question is: how should one teach mathematics today? Admitting the role of technology in shaping mathematics, accepting that the computer is going to play an increasingly important role in the future, and admitting that formal mathematics is not quite suited to computers, the conclusion seems to be forced that a different type of mathematics should be taught. The calculus is at the core of many numerical computations, but can one at all do the calculus without real numbers? An alternative mathematical epistemology could be invented *ab initio*. Or one could fall back on the alternative epistemology of mathematics in India, as described in the *Yuktibhâsa*. This alternative epistemology provided the natural soil in which the calculus grew. Recognizing the existence of this alternative epistemology of mathematics requires, however, an alternative account of the history of mathematics. This is an illustration of the general maxim that the history of mathematics has profoundly influenced its philosophy, so that to change the philosophy of mathematics, one must also revise its history. A condensed account of the suggested revision follows.

According to the Western history of the calculus, the calculus was the invention of Leibniz and Newton, particularly Newton, who used it to formulate his 'laws' of physics. In a series of papers, I have pointed out that this narrative needs to be significantly changed for several reasons.

(a) The key result of the calculus, attributed variously to Gregory,[58] Newton, and to Newton's student Brook Taylor,[59] is the infinite-series expansion today commonly known as the Taylor's series expansion. This infinite series expansion is found in India a few centuries before Newton in the work of Madhava of Sangamagrama and in the later works like Nilkantha's *TantraSangraha* (1501 CE), Jyeshtadeva's *YuktiBhâsa* ("Discourse on Rationale" c. 1530 CE)[60] the *TantraSangrahaVyâkhya*, the *YuktiDîpikâ*, the *Kriyâkramakari*, the *KaranaPadhati* and other such widely distributed and still existent works of what has been called the Kerala school of mathematics and astronomy.

निहत्य चापवर्गेण चापं तत्तत् फलानि च ।
हरेत समूलयुग्वर्गैस्त्रिज्यावर्गाहतैः क्रमात् ॥
चॉप फलानिं चाधोऽधौ न्यस्योपर्युपरि त्यजेत् ।
जीवाप्त्यै ...

This key passage may be translated as follows.

> Multiply the arc by the square of the arc, and repeat [any number of times]. Divide by the product of the square of the radius times the square of successive even numbers increased by that number [multiplication being repeated the same number of times]. Place the arc and the results so obtained one below the other and subtract each from the one above. These together give the *jîvâ*…

Jîvâ relates to the sine function. Etymologically, the term sine derives from *sinus* (= fold) a Latin translation of the Arabic *jaib* (opening for the collar in a gown), which is a misreading of the Arabic term *jîbâ* (both terms are written as *jb*, omitting the vowels). Mathematically, however, as is well-known, *jîvâ* and *sara*, like the sine and cosine of Clavius' sine tables (as their very title shows),[61] were not the modern sine and cosine but these quantities multiplied by the radius r of a standard circle. The *jîvâ* corresponds to $r \sin \theta$, while the *sara* corresponds to $r\,(1 - \cos \theta)$.

In current mathematical terminology, this passage says the following. Let r denote the radius of the circle, let s denote the arc and let t_n denote the nth expression obtained by applying the rule cited above. The rule requires us to calculate as follows. (1) Numerator: multiply the arc s by its square s^2, this multiplication being repeated n times to obtain $s \cdot \prod_1^n s^2$. (2) Denominator: Multiply the square of the radius, r^2, by $[(2\,k)^2 + 2\,k]$ ("square of successive even numbers increased by that number") for successive values of k, repeating this product n times to obtain $\prod_{k=1}^{n} r^2\,[(2k)^2 + 2k]$. Thus, the nth iterate is obtained by

$$t_n = \frac{s^{2n} \cdot s}{(2^2 + 2) \cdot (4^2 + 4) \cdot \ldots \cdot [(2n)^2 + 2n] \cdot r^{2n}}$$

The rule further says:

$$jîvâ = (s - t_1) + (t_2 - t_3) + (t_4 - t_5) + \ldots$$

Substituting:

(1) *jîvâ* $\equiv r \sin\theta$,
(2) $s = r\,\theta$, so that $s^{2n+1}/r^{2n} = r\,\theta^{2n+1}$, and noticing that
(3) $[(2k)^2+2k] = 2k\cdot(2k+1)$, so that
(4) $(2^2+2)(4^2+4)\ldots[(2n)^2 + 2n] = (2n+1)!$,
and cancelling r from both sides, we see that this is entirely equivalent to the well-known expression

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots.$$

This verse is followed by a verse describing an efficient numerical procedure for evaluating the polynomial.[62] The existence of these verses has been known to Western specialists for nearly two hundred years, and is today acknowledged in some Western texts on the history of mathematics, like those of Jushkevich,[63] Katz[64] etc.

In current mathematical terminology, the key step in the *Yuktibhâsâ* rationale for the above series is that

$$(1)\quad \lim_{n\to\infty} \frac{1}{n^{k+1}}\sum_{1}^{n} i^k = \frac{1}{k+1},\ k = 1, 2, 3,\ldots,$$

in the sense that the remaining terms are numerically insignificant, for large enough n.

(b) A relevant epistemological question is this: did Newton at all understand the result he is alleged to have invented? Did Newton have the wherewithal, the necessary mathematical resources, to understand infinite series? As is well known, Cavalieri in 1635 stated the above formula as what was later termed a conjecture. Wallis, too, simply stated the above result, without any proof.[65] Fermat tried to derive the key result above from a result on figurate numbers, while Pascal used the famous “Pascal’s” triangle[66] long known in India and China. Though Newton followed Wallis, he had no proof either,[67] and neither did Leibniz who followed Pascal. Neither Newton nor any other mathematician in Europe had the mathematical wherewithal to understand the calculus for another two centuries, until the development of the real number system by Dedekind.

(c) The next question naturally is this: if Newton and Leibniz did not quite understand the calculus, how did they invent it? In the amplified version of the usual narrative, how did Galileo, Cavalieri, Fermat, Pascal, and Roberval etc. all contribute to the invention of a mathematical procedure they couldn’t quite have understood? The frontiers of a discipline are usually foggy, but here we are talking of a gap which is typically 250 years.

(d) Clearly a more natural hypothesis to adopt is that the calculus was not invented in Europe, but was imported, and that the calculus took nearly as long to assimilate as did zero. Since authoritative Western histories of mathematics are replete with wild claims of transmission from Greece, an appropriate standard is needed for the evidence for transmission. I have suggested that we follow the current legal standard of evidence, by establishing (i) motivation, (ii) opportunity, (iii) documentary evidence, and (iv) circumstantial evidence.

Motivation (a) : Europe had strong motivation to import mathematical and astronomical knowledge in the 16th and 17th centuries CE, because mathematics and astronomy were widely regarded as holding the key to navigation which was the route to prosperity hence the critical technology of the times. As is now widely known, Europe did not have a reliable technique of navigation, and European governments kept offering huge prizes for this purpose from the 16th until the 18th century CE. Indeed, the French Royal Academy, the Royal Society of London etc. were started in this way in an attempt to develop the astronomical and mathematical procedures needed for a reliable navigational technique.

The first navigational problem concerned latitude: right from Vasco da Gama, Europeans attempted to learn the Indo-Arabic techniques of determining latitude through instruments like the Kamâl. The Indo-Arabic technique of determining latitude in daytime assumed a good calendar, and this led to the Gregorian calendar reform. As a student and correspondent of Pedro Nunes, Clavius presumably understood that reforming the calendar, and changing the date of Easter was critical to the navigational problem of determining *latitude* from the observation of solar altitude at noon, as described in widely distributed Indian mathematical-astronomical texts, and calendrical manuals.

Opportunity: On the other hand, right from the 16th century there was ample opportunity for Europeans to collect Indian mathematical-astronomical and calendrical texts. The Jesuits were in India, with their strongest centre being Cochin, from where a copy of the *Tantrasangraha* or *Yuktibhâsâ* could easily have been procured. Each Jesuit was expected to know the local language, and Alexander Valignano declared that it was more important for the Jesuits to know the local language than to learn philosophy. They could hardly have functioned without a knowledge of the local calendar and days of festivity. One of the earliest Jesuit colleges was at Cochin, and it typically had an average of about 70 Jesuits during the period 1580–1660. Prior to this period, printing presses had already been started in languages like Malayalam and Tamil, and Malayalam was being taught at the Cochin college at the latest by 1590.

Documentary evidence: Moreover, the Jesuits were systematically collecting and translating local texts and sending them back to Europe. In particular, Christoph Clavius, head of the Gregorian Calendar Reform Committee changed the mathematics syllabus of the Collegio Romano, to correct the Jesuit ignorance of mathematics, and from the first batch of mathematically trained Jesuits he sent Matteo Ricci to Cochin to understand the available texts in India on the calendar, and the length of the year.[68]

Motivation (b): Pedro Nunes was also concerned with loxodromic curves, the key aspect of Mercator's navigational charts, which involved a problem equivalent to the fundamental theorem of calculus. Pedro Nunes obtained his loxodromic curves using sine tables, which tables were later corrected by Christoph Clavius and Simon Stevin. Thus, precise sine values were a key concern of European astronomers and navigational theorists of the time. The infinite series expansion as used by Madhava to calculate high-precision sine values, the coefficients used for efficient numerical calculation of these values, and the 24 values themselves were incorporated in a single sloka each, the last two found also in the widely distributed calendrical manuals like *Karanapadhati*.

Motivation (c): Europeans could not use Indo-Arabic techniques of longitude determination because of a goof-up about the size of the earth. Columbus, to promote the financing of his project, downgraded the earlier accurate Indo-Arabic estimates of the size of the earth by 40%. But this size entered as a key parameter in the Indo-Arabic techniques. Nevertheless, Europeans remained interested in the Indo-Arabic techniques of longitude determination, and when the French Royal Academy ultimately developed a method to determine longitude on land, it was a slight improvement of the technique of eclipses mentioned in the texts of Bhaskara-I, and the tome of al Biruni.

Circumstantial evidence: Once in Europe the imported mathematical techniques could easily have diffused, and there is circumstantial evidence that many contemporary mathematicians knew something of the material in Indian texts. For example, Clavius' competitor and critic Julian Scaliger introduced the Julian day-number system, essentially the *ahârgana* system of numbering days followed in Indian astronomy since Aryabhata. Galileo's access to Jesuit sources is well documented, as is that of Gregory and Wallis. Cavalieri was Galileo's student, and Gregory does not claim originality for his series. Marin Mersenne was a clearinghouse for mathematical information, and his correspondence records his interest in the knowledge of Brahmins and 'Indicos'. Fermat, Pascal, Roberval were all in touch with him, and part of his discussion circle. There is other circumstantial evidence to connect Fermat to Indian mathematical texts, for instance his famous challenge problem to European mathematicians, and particularly Wallis, involves a solved problem in Bhaskara's *Beejganita*.[69] 'Julian' day-number, "Fermat's" challenge problem, and "Pascal's" triangle cover only some of the circumstantial evidence of the inflow of mathematical and astronomical knowledge into Europe of that period, but I will not examine more details here, since I regard the above as adequate to make a strong case for the transmission of the calculus from India to Europe in the 16th and 17th c. CE.

4 Mathematics Education

To jump from the past to the future: what bearing do these concerns have on current mathematics education? In the light of the revised history of the calculus, in the light of the argument that mathematics is a social construction that is likely to change with changing technology, especially the widespread use of computers, how should mathematics and calculus be taught today?

In accordance with the principle that *phylogeny is ontogeny*, the natural way to learn the subject is to retrace its ontogenesis. The current way of teaching the calculus retraces the ontogenesis of the calculus in Europe. The calculus is first taught as an intuitive and unclearly understood thing, which is nevertheless indispensable for practical purposes. After at least a couple of years (representing the gap of a couple of centuries in Europe), one teaches the real number system, and the elements of mathematical analysis, and the Riemann integral, finally leading to a proof of the so-called Taylor's theorem, the classical version of the fundamental theorem of calculus, and Peano's existence theorem for the solution of differential equations etc. Numerical analysis, and discretisation, is typically expected to come *after* this. Since pedagogy follows the (perceived) ontogeny, the revised ontogenesis suggests a revised way to teach mathematics. The 'numerical calculus' of the *Yuktibhâsâ*, as distinct from both calculus and analysis, can be taught directly as a technique of computation, using floating point numbers and empirical rationale.

A similar conclusion follows from the argument that formal mathematics is a social construction, likely to change with technology. The computer has enormously simplified complex calculations, and has thus encouraged the view of mathematics as calculation. By encouraging the idea of mathematics-as-calculation, computer technology has already created sharp conflicts with Western mathematical orthodoxy, and its theological orientation towards mathematics-as-proof. Ideally one is expected to prove a convergence theorem for an algorithm before writing a computer program for it. Ideally one should even prove the program that one uses: of what value is a computer-aided proof of the four-color theorem if the program used in the proof cannot itself be proved? This requirement of proof is rarely respected in practice. Few people who use computers (physicists, engineers etc.) have enough mathematical training to provide these kinds of proofs. Even if they have, the required proofs may simply not be available, as in the case, mentioned earlier, of stochastic differential equations driven by Lévy motion. A practical requirement must be met here and now. For a practical requirement, one generally cannot wait for as long as one may be ready to wait to demonstrate the validity of an eternal truth.

Both arguments suggest that it is time to revise the mathematics syllabus outlined by Plato.

(a) Mathematics-as-calculation should be taught for its practical value, at the elementary and intermediate level. This applies especially to the calculus: given its revised ontogenesis, and given its implementation on computers.

(b) Mathematics must be taught as empirically based, and fallible. Thus, certainly, the question no longer is: *what* is the value of 1–1+1–1+1–1...? Nor is it any longer the question: *how* should one define 1–1+1–1+1–1... so as to lead to a theory most acceptable to authoritative mathematicians? Rather, the question is this: are there methods of summing this series that are empirically useful? Hence, a technique of calculation, e.g. 1–1+1–1+1–1... = ½, could be acceptable if it is of practical value, like an engineering technique, or can be empirically validated, like a physical theory, or in conjunction with a physical theory. A given technique of calculation may be fallible, and may not work in another case: for example, the standard technique of extracting a finite value from a divergent integral, as used in renormalization in quantum field theory, does not work with shock waves. While one need have no qualms about non-universality, naturally, the most convenient conventions will be those that are most widely applicable.

(c) On the other hand, I feel Proclus did have a point, that at least at an elementary level, mathematics-as-proof does afford a certain aesthetic satisfaction, even if mathematics as proof does not fulfill the original promise of providing secure knowledge. Thus, I feel that the teaching of mathematics-as-proof, like the teaching of music, or other art form, ought not to be discontinued altogether, but it should be an optional matter, which could be taken up, especially at higher levels, by those interested in it.

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