1 Mathematics and Culture

IN a recent essay "On the Nature of Linkages between Science, Technology, Philosophy, and Culture," Prof. D. P. Chattopadhyaya[1] has argued that "even the abstract disciplines like mathematics are influenced by practical and social considerations". In the sequel, he has briefly pointed to the relation of geometry to architecture and surveying, and of *Śūnya* to *Śūnyavāda*. He has referred to the ontological status of infinities and infinitesimals, to the debate between intuitionists and formalists, and has concluded this particular argument (embedded as a strand in a wider argument) with quotations from John von Neumann and Norbert Wiener.

I should mention prefatorily that Prof. Chattopadhyaya's choice of mathematicians from whom to quote is not incidental. He clearly sees philosophy as a process of integrating experience into a unified whole: the wider and more unique the experience, the greater the paradoxes, and the richer will be the philosophy. And John

[1] D. P. Chattopadhyaya, *Science, Technology, Philosophy and Culture,* PHISPC, 1996, 292–318.

von Neumann and Norbert Wiener are easily identified as among the most catholic of mathematicians.

I accept the thesis that mathematics is influenced by practical and social considerations, and I have had occasion to comment elsewhere on these influences in the context of (a) the logic underlying inference, (b) geometry, and (c) *Śūnya*, and infinities and infinitesimals[2] In the following, I will collect together these comments in the hope that collecting the arguments in one place will further the basic thesis of the influence of social and cultural considerations on mathematics.

1.1 Industrial capitalism and mathematical authority

Let me begin with the more general part of the thesis: the influence of industrial capitalism on the belief in the universality of mathematics.

At the present moment, why does the state support mathematicians? Is this a matter of charity, or does the state derive (or expect to derive) some benefit from this? The logic is quite clear: mathematics is a key input to modern science, which is a key input to technol-

[2] C. K. Raju, *The Eleven Pictures of Time* (submitted). "On the Mathematical Epistemology of Śūnya" (to appear) Proceedings of the Seminar on the Concept of Śūnya, INSA and IGNCA, New Delhi, Feb 1997. "How should Euclidean geometry be taught in schools?" (to appear) Proceedings of the Conference on History and Philosophy of Science. . . , International Workshop on the History of Science: Implications for Science Education, Homi Bhabha Centre for Science Education, Bombay, Feb 1999.

ogy, which is the key to economic and physical domination.

Present-day mathematics has grown along with modern science under conditions of industrial capitalism. The substantial increases in profit come from technological innovation; consequently the scientist must have a single-minded focus on innovation useful for commercial production—when he is not working like von Neumann on designing new weapons like the atomic bomb, used to extract surplus by other means. Innovation has, thus, become a commodity, and specialisation boosts the efficiency of production of commodified innovation; hence most scientists tend to be very specialised. One consequence of this is that scientists are not able to understand each other or communicate with each other. If a mathematician has to read a paper not exactly in his field, this process could easily take a determined effort lasting for a year or two. With such formidable difficulties in communication, scientists quickly start relying on authority. This is the first consequence of industrial capitalism: because it hopes to profit from specialisation, it encourages reliance on authority.

Thus, the new standard of truth is this: if it is published by an important person in a respectable journal it must be true or, at any rate, very likely true (though there is still the possibility of a small error somewhere if one is speaking of the four-color theorem, or Fermat's last theorem). The most pathetic example of this standard of truth is the grievous mathematical error[3] in

[3] C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, 1994, Chapter 5b. The error is that the essential history-

a paper published by Einstein[4] in the *Annals of Mathematics*, in 1938, on the relativistic many-body problem, which exposes his fundamental lack of understanding of the special theory of relativity relative to Poincaré.

There is another reason why the prevalent social conditions systematically encourage the process of deciding truth by authority. Barring a few hundred relativists, and perhaps a few thousand people who might have some idea of it, most people in the world would be unable to judge for themselves the truth of the above example about Einstein. This state of affairs is not incidental. Commodified innovation is produced by scientists through a process of research; hence, the state is willing to invest resources into research facilities that scientists need to produce innovation. The state also does invest in the education of scientists, but only with the objective of reproducing the scientific labour power needed to produce innovation. It is well understood why, under conditions of industrial capitalism, there is systematically greater investment in production than in reproduction of the labour consumed in production. Hence, there is a systematic bias in the state support for science: more resources are invested in research facilities than in education. (In particular, the state is no longer interested in enabling people through education to understand the world around them. Not only has education been delinked from the needs of theol-

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dependence of the relativistic many-body problem has been wished away by using a Taylor expansion in powers of the delay to convert a retarded functional differential equation into an ordinary differential equation.

[4]A. Einstein, L. Infeld, and B. Hoffman, *Ann. Math.* **39** (1938) 65.

ogy, but "understanding" is something that most scientists look down upon as "philosophy", since it consumes the time that could be more actively spent in the process of engineering useful innovations.) As a result of this systemic bias against education, in the state support for science, most people are scientifically illiterate, even in the developed countries[5] In the interaction of illiterate patients with doctors one can easily see how illiterate persons are left with no option but to decide truth by relying on authority, whether that authority is conferred by the media or the state. This is the second consequence of industrial capitalism: it encourages reliance on authority by creating widespread scientific illiteracy or information poverty. (Spengler had already anticipated this widespread scientific illiteracy as a process contributing to the decline of the West.)

One can also enquire into the nature of this authority: what bearing does it have on truth? How reliable is authority, on an average? To continue the analogy, the illiterate patient has no option but to trust the doctor, but even to a casual observer it is obvious that a medical career is much sought after not out of a widespread desire to help out humanity at large, but to enable the person to lead a good life, as it is defined under industrial capitalism. The doctor's first concern usually is extraction of surplus rather than the health of the patient, and this is especially true if the patient is illiterate and hence not very important. Consequently, the doctor's prescription may suit the health of the pharmaceutical company more than that of the patient. Unlike

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[5]Gerald Holton, *Science and Anti-Science*, Harvard University Press, Cambridge, Mass, 1994, p 147.

doctors, scientists who are in authority are necessarily employed by, hence dependent upon, state and private capital.

Second, industrial capitalism is a great uniformizer, because standardization is essential for mass production. Once something becomes a standard, market logic tends to drive out others: a publisher will be more willing to publish a text in mathematics rather than a monograph on intuitionism. This process relies, like the market, on statistical effects, rather than any absolute prohibition: difference is not prohibited, but is made so disadvantageous that few people care to differ. Consequently, those in authority do not differ too much from each other. Thus, industrial capitalism encourages a process of uniformity and standardization in opinion.

The above processes lead to the remarkably widespread agreement that sociologists have observed among practitioners of mathematics and science. But this uniformity and standardization of opinion ought not to be mistaken for universality as it often is. In the context, uniformity of opinion does not make the opinion itself more reliable: if a variety of doctors prescribe the same drug this does not mean that that drug is most suited to one's health, it might simply mean that this is a drug being vigorously promoted.

To my mind it would be facile to set aside the above observations, regarding the determination of mathematical and scientific truth through authority, as concerning *practice* rather than *principle*. It is a myth that principles are insulated from practice. The very same practical and social considerations may infiltrate not only the allegedly universal and metaphysical "truths" of mathematics but also the very principles used to decide these truths—principles that have been and can only be formulated by authoritative mathematicians. If practical considerations can penetrate to the content of relativity, there is no reason why they cannot penetrate the content of the philosophy of science or mathematics. We will see this in greater detail below. Since these principles, as currently articulated in the formalistic philosophy of mathematics, have no external empirical anchor, it is all the more important to recognize the social processes within which mathematical authority is anchored.

Thus, authority decides mathematical truth—the veracity of mathematical theorems and the principles used to decide this veracity. The obvious point about authority as the standard of mathematical truth is that authority is socially conferred. It would, of course, be excessively naive (or religious) to imagine that social processes are such that they automatically (or by design) "select the fittest" and confer authority on those who seek truth through creative insights. Thus, it is not only in present-day India that knowledgeability and creativity have little relation to scientific authority. The primary interest under industrial capitalism is neither in understanding nor in the creative process of innovation, but in *control* of information or the ownership of the innovated commodity, as decided by patents, authorship of papers etc.[6] As a clerk in the patent of-

[6]This is reflected in the social phenomenon where many heads of scientific establishments routinely claim ownership of innovation by attaching their names to papers they may never have read,

fice, Einstein understood the subtler legalities of this process: that one may copy ideas if one does not copy the expression verbatim. A more recent example of this sort is Bill Gates, one of the richest men of all time, who legally won the claim of having innovated the windowing software that, despite its bugs, bears a striking resemblance to the earlier software of Apple Macintosh. The relative unimportance of the creative process is emphasized by the fact that no one has heard of the person who initially thought up the point-and-click concept behind the windowing software. Authority flows from ownership, and ownership, laws regarding ownership, and the principles on which these laws are based, are all rooted in social processes that it is not necessary to go into here.

To recapitulate, under industrial capitalism social processes tend to decide mathematical truth in two steps. (a) Overspecialisation of scientists, and widespread scientific illiteracy of others, both, strongly encourage reliance on authority, and (b) authority devolves on those who are better able to manipulate social processes of deciding ownership of innovation rather than on those who are most knowledgeable or innovative—there is also a systematic decline of the best!

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and may not even be able to understand or explain. Conversely, many younger scientists seek to gain authority by promoting this practice. The Darcy case demonstrated that this sort of thing is systematically true of leading institutions around the world. See, W. W. Stewart and N. Feder, "The integrity of scientific literature," *Nature* **325** (1987) 207–214.

1.2 Metamathematics, logic, and geometry

These general considerations make it plausible that social and cultural processes may infiltrate the principles and practices of mathematics that is widely accepted today as embodying universal truths. Let us now turn to specifics.

As already observed, deciding truth by authority is particularly well-suited to 20th century formalistic mathematics, which sees mathematics as *a priori*, and pre-empirical, and consequently has no external reference point, such as the empirical world, from which authority may be refuted. According to the abstract criteria specified by this philosophy, a mathematical statement is valid if it is a theorem. A theorem in the present-day sense is the last sentence of a proof. A proof is a sequence of statements such that each statement is either an axiom or is derived from two preceding statements by the use of modus ponens or other specified rule of reasoning[7]

But what guarantees the validity of the criteria of mathematical validity? Why should mathematics be *a priori* and divorced from the empirical? Is this a cultural belief, or is this universally the case. The slightest acquaintance with the history of mathematics tells us that the above definition of proof originated with Hilbert's metamathematics. That metamathematics, in turn, relates to a particular idealisation of the demon-

[7] E. E. Mendelson, *Introduction to Mathematical Logic*, The University Series in Undergraduate mathematics, Van Nostrand Reinhold, New York, 1964, p 29.

strations in the *Elements*, long regarded, on theological grounds, by Arabs and the West as the ultimate standard of demonstration. (The theological grounds were, of course, different in the two cases: among the Mut'azilah it was the deduction of manifest truth from the postulated "equality" in the *Elements* that was important, while among the schoolmen the important thing was rational demonstration of universal truths to unbelievers who did not accept the authority of the scripture.) The social and cultural factors implicit in the current definition of mathematical truth can therefore be brought out by an analysis of elementary geometry.

The interplay of these factors is well brought out in the following quotation from an elementary Indian school text[8] begins its exposition of Euclidean geometry as follows.

"The *Baudhayana Sulbasutras*...contains [sic] a clear statement of the so-called Pythagoras theorem. The proof of this theorem is also implicit in the constructional methods of the *Sulbasutras*."

The first part of the first sentence of this quotation is correct. The Baudhāyana *sulba sūtra*-s do contain a statement semantically very similar to the proposition 1.48 of the *Elements*[9] So do the other *sulba-sūtra*-s[10]

[8]A. M. Vaidya et al, *Mathematics. A Textbook for Secondary Schools*, Class IX, NCERT, 1998, p 123–124.

[9]T. L. Heath, *The Thirteen Books of Euclid's Elements*, Dover, New York, 1956, Vol. 1.

[10]Baudhāyana 1.12–13, Apastamba, 1.4, Kātyāyana, 2.7, Mānava, 10.10. S. N. Sen and A. K. Bag, *The Sulba Sūtras of Baudhāyana, Apastamba, Kātyāyana and Mānava,* with Text, English Translation and Commentary, INSA, New Delhi, 1983, p 151.

This statement was constantly used in the *sulba-sūtra*-s. It was also the basis of trigonometry in the Indian astronomical tradition. However, the validity of the rest of the quotation is very doubtful. In their eagerness to claim ownership through historical priority, the authors of the text have incorrectly sought to relate incommensurate forms of mathematics originating in diverse cultures.

The statement in the various *sulba sūtra*-s is NOT a theorem in the sense of present-day mathematics. There cannot be a theorem without a proof, and there is no proof in the present-day sense specified above, not even an attempt at proof. The statement was regarded as true, but no proof was stated. As in the story of the Epicurean ass, this could make all the difference. (To the Epicurean charge that any ass knows that proposition 1.20 of the *Elements* is true, Proclus responded that only the mathematician knows why it is true, because he has a proof.)

Because no proof was stated it does not, of course, follow that the authors of the *sulba sūtra*-s did not know why the result was true. But the method of proof that convinced them may well have differed from the current definition of proof. Thus, it is incorrect to assert that the constructional methods used in the *sulba-sūtra*-s implicitly lead to a proof in a formalistic sense. It is incorrect because the rationale for the formula for a right-angled triangle, from the constructional methods of the *sulba-sūtra*-s right down to the 16th century *Yuktibhāṣā,* explicitly appeals to the empirical. Indeed, the *Yuktibhāṣā* is the only full-fledged text on rationales to have been translated till now, and, as a prelude to the infinite series used in squaring the circle, this text justifies the formula for a right-angled triangle as follows[11] (See figure).

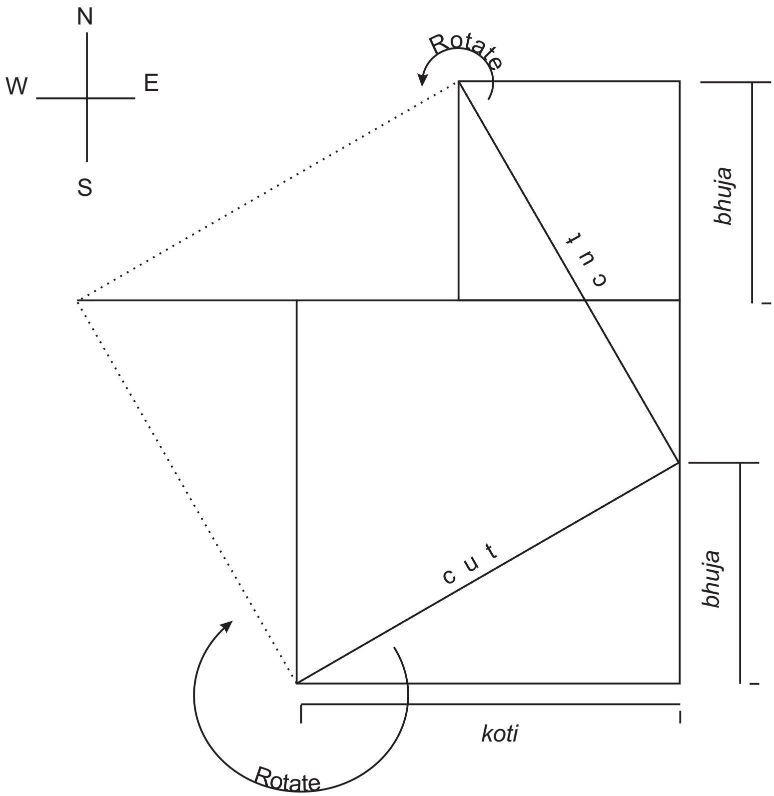

Figure 1.1: The *Yuktibhāṣā* proof of the "Pythagorean" theorem.

> "Now, draw a square [with its side] equal to the *koṭi* [longer side of the triangle], and

[11] K. V. Sarma (ed. and Tr.), *The Gaṇitayuktibhāṣā of Jyeshṭadeva* (unpublished), Chapter 6, The Circle, p 36–37.

another equal to the *bhuja* [shorter side of the triangle]. Let the *bhuja* square be on the northern side and the *koṭi* square on the southern side, in such a way that the eastern side of both the sides [squares] falls on the same line, and in such a manner that the southern side of the *bhuja*-square lies alongside the *koṭi*. [Since the *koṭi*] is longer than the *bhuja* on the [*koṭi*] side, there will be an extension [of the *koṭi*] towards the western side further than the *bhuja*. From the northeast corner of the *bhuja*-square, measure southwards upto the *koṭi*, and mark [the spot] with a point. From this [point] the line towards the south will be of the length of the *bhuja*. Then cut on lines from the point to the southwest corner of the *koṭi*-square and the northwest corner of the *bhuja*-square, dividing the squares [into equal triangles]. Allow a little clinging at the two corners so that the cut portions do not fall away. Now break off the two parts [i.e., the triangles] at the point, turn them round alongside the two sides of the bigger (i.e., *koṭi*) square, so that they meet at the north-east, and join them, so that the inner cut of one joins with the outer cut of the other. The figure formed thereby will be a square. And the side of this square will be equal to the hypotenuse of the original *bhuja-koṭi*[rectangle]. Hence it is established that the sum of the squares

> of the *bhuja* and *koṭi* is equal to the square of the *karṇa* [hypotenuse]...."

Superficially, we can observe that, in the above proof, the figure is meant to be drawn on a palm-leaf, portions of which are to be cut and moved about in space—all these are processes not permitted in Hilbert's reconstruction of geometry. That reconstruction incorporates the doubt that moving figures about in space may deform them, like movement over an uneven surface may deform a shadow; prescribing rules for such movement, as in parallel propagation of Riemannian geometry lead to a different geometry. Moreover, even *drawing* figures may be misleading. Points, lines, and triangles in Hilbert's geometry are abstract objects; there is no sanctity to the representation of points by dots. In his famous 'beermugs' remark, reportedly inspired by Wiener, Hilbert stressed the model-indifference of axioms: points, lines and planes could well be represented by chairs, tables, and beermugs. (More recently, people have been seriously debating whether this ontologically committed Hilbert to the existence of an infinity of [full] beermugs!) Hence also it would be erroneous to identify the points of axiomatic theory with their conventional representation. This point of view must be clearly distinguished from that of the *Elements*, where, in contrast to modern axiomatic geometry, the conventional representation at least approximates to the truth: a dot approximates the idea of a point as that which has no part, and a conventional line still approximates the idea of a line as a breadthless length, while beermugs would be quite out of place. The conventional representation approximated the semantics of the *Elements*, formal theory is pure syntax devoid of semantics.

In contrast to formalistic mathematics, the key thing to observe is that the *Yuktibhāsā* proof appeals to the manifest (*pratyaksha*); it corresponds to a proof in the sense of Nyāya. It is manifestly possible to draw a triangle on a palm leaf, and to cut and move about portions of it in space. What is manifest, needs no further proof. Neither can the manifest be semantically void; it ostensively admits meaning. Nevertheless, the manifest is rejected by Hilbert. This process of moving figures about is the basis also of measurement, the possibility of which is completely denied in Hilbert's synthetic reconstruction of the geometry of the *Elements*[12] Consequently, Hilbert changes Theorem 4 of the *Elements* into the Side-Angle-Side postulate. The NCERT's ninth standard text, does the same, failing to observe how this contradicts its earlier statement quoted above.

The present-day notion of proof is completely divorced from the empirical. Present-day mathematics is *a priori* and non-empirical, perhaps even anti-empirical, though it may be used in a physical theory which models the empirical world. This is in line with Platonic philosophy which regarded mathematics as perfect and the real world as imperfect. Hence, there had to be different criteria of proof for mathematics and physics.

This divorce from the empirical creates the possibility of having a mathematical theory that is physically absurd. Consider, for example, the following formal

[12]Birkhoff's metric reconstruction, however, permits measurement, while the *sulba*, being flexible, permits measurement also of the length of curved lines.

theory: an algebraic system with the symbols 1, 2, 3, 4, 5, and a binary operation + defined by the following odd-looking rules for addition.

$1 + 1 = 3$, $1 + 2 = 4$, $1 + 3 = 5$,$1 + 4 = 1$, $1 + 5 = 2$, and etc. Instead of writing out the full table, I will indicate the private shortcut: the general rule is given by $a+b = a+b+1$, where the + sign on the right refers to usual addition modulo 5.

(It is clear how to extend this rule to define addition and multiplication allowing for an infinity of symbols.) In this algebraic system we would have the following

THEOREM. $2 + 2 = 5$.

PROOF. Follows directly from the definition for +.

$2 + 2 = 5$ is a valid theorem within the above algebraic system. One cannot even dismiss the above system on the ground that it lacks any practical application: for there is the age old argument that it may find a practical application tomorrow. (The classic instance is prime factorization, which has finally found a practical application in everyday methods of encryption and decryption used today in secure email.) The best one can do is to express one's like or dislike for the system, and the seriousness with which one's opinion is taken would depend upon the authority one has.

The question now is this: would an orthodox Nayyāyikā have accepted the above Theorem as a valid mathematical result? I regard this as doubtful, for the Nayyāyikā might demand an empirical example where $2 + 2 = 5$. However, I will leave this question open for Nayyāyikā-s to answer. It should, nevertheless, be pointed out that no early school of thought in the Indian tradition would have readily accepted the denial of the manifest.

Apart from the question of the Platonic rejection of the empirical in a mathematical proof, there is the other question of the logic underlying inference. This difference concerns not Nayyāyikā-s but Buddhists and Jains. Hilbert took for granted that logic must be 2-valued, and Aristotelian, just because there was complete unanimity among his contemporaries on this matter.

Historically, how did this unanimity arise? An examination of the history of Western philosophy from schoolmen to the Renaissance shows clearly the continuing influence of Aristotle and Plato. The current beliefs about logic can be traced back to the beginnings of scholastic philosophy. The beginnings of scholastic philosophy, as is well known, were deeply influenced by Arabic learning. Apart from the Averroists, schoolmen like Duns Scotus took up positions that are strongly reminiscent of al Ghazāli. But Ibn Sīnā, al Ghazāli and Ibn Rushd all accepted Aristotelian logic, with even al Ghazāli explicitly asserting[13] that Allah was bound by the laws of logic. The belief in the universality of (Aristotelian) reason within a universal state led by a universal church also found its first clear articulation in the writings of Roger Bacon.

To reiterate, uniformity of opinion is not the same as universality. Plato and Aristotle are not universally important across space and time. The belief in a truth-functional 2-valued logic was denied by the Buddhists

[13]*Tahāfut al Falāsifa*. (Tr) S. A. Kamali, *Al Ghazali's Tahafut al-Falasifa*, Pakistan Philosophical Congress, Lahore, 1958, p 189 and p 194.

and Jains. Buddhists had a logic of "4 alternatives". In the Brahmajāla Sutta[14] the Buddha describes four wrong views about the world, whose adherents argue as follows.

(1) The world is finite.
(2) The world is not finite.
(3) The world is both finite and infinite.
(4) The world is neither finite nor infinite.

The semantic interpretation of (3) is that the world is finite up-and-down and infinite across. The semantic interpretation of (4) is that all three of the preceding views are wrong; it is said to be "hammered out by reason". A fifth possibility was explicitly denied. Later on in the same Brahmajāla Sutta, the Buddha again rejects the use of more than four possibilities, describing them by the epithet: the "Wriggling of the Eel"[15]

Nagarjuna also has the famous tetralemma putting forward the proposition[16]

Everything is
such
not such
both such and not such
neither such nor not such.

[14]*Dīgha Nikāya*. Tr. Maurice Walshe, *The Long Discourses of the Buddha. A Translation of the Dīgha Nikāya*, Wisdom Publications, Boston, 1995, pp 78–79.

[15]*Dīgha Nikāya*. Tr Maurice Walshe, cited above, pp 80–81.

[16]*Mūlamādhyamakakārika* 18.8, Sanskrit text and Eng. (Tr) David J. Kalupahana, *Nagarjuna. The Philosophy of the Middle Way*, SUNY, New York, 1986, p 269.

The writings of Dinnaga[17] on this point are a bit obscure, particularly because a key work (*Hetucakra*; "Wheel of Reason") is preserved only in the Tibetan, and in the works of Nayyāikā opponents, and there seems to be a serious difference of opinion regarding its translation—a point on which I am not qualified to comment. I accept that Dinnaga had introduced logical quantification, but I feel this alone cannot explain everything, unless we grant that the quantification was based on a non-Aristotelian logic. In this connection, I would like to point to the last stanza of the *Hetucakra*.

Matilal[18] accepted that the standard negation does not fit Buddhist logic. G. N. Ramachandran[19] suggested that Buddhist logic corresponds to a cyclic negation for an 8-valued logic. On the other hand, Hal-

[17]Tr. D. Chatterji, "*Hetucakranirnaya*", *Indian Historical Quarterly* **9** (1933) 511–514. Reproduced in full in, R. S. Y. Chi, *Buddhist Formal Logic*, The Royal Asiatic Society, London, 1969, reprint Motilal Banarsidass, Delhi 1984. Chi objects to the exposition of Vidyabhushan (cited below).

[18]B. K. Matilal, *Logic, Language, and Reality*, Motilal Banarsidass, Delhi, 1985, p 146: "My own feeling is that to make sense of the use of negation in Buddhist philosophy in general, one needs to venture outside the perspective of the standard notion of negation." See also, H. Herzberger, Double Negation in Buddhist Logic", *Journal of Indian Philosophy* **3** (1975) 1–16.

[19]G. N. Ramachandran, preprint, Indian Institute of Science, Bangalore, 1988(?).

dane's[20] interpretation of Bhadrabahu's[21] *syādavāda* is readily seen to correspond to the semantics of a three-valued logic. My own reading is that the underlying Buddhist logic is quasi truth-functional as I believe to be true also of quantum logic[22] for it is only in this case that one can meaningfully assert that both A and not-A hold.

The point of bringing in quantum logic is this: if one does eventually decide to appeal to the empirical, in support of logic, a 2-valued logic need not be the automatic choice. One might perhaps want to start with a quantum logic as the empirical basis of mathematics, so that *no* conclusion could be drawn from the statement that Schrödinger's cat is both dead and alive. (In conventional logic, *any* conclusion could be drawn from this statement.)

In any case, there is no case for the "universality" of the logic underlying present-day mathematics and metamathematics. If the logic underlying modern-day formalistic mathematics were to be changed, that would, of course, change also the valid theorems or universal

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[20] J. B. S. Haldane, "The Syadvada system of Predication", Sankhya, Indian Journal of Statistics, 18 (1957) 195. Reproduced in D. P. Chattopadhyaya, *History of Science and Technology in Ancient India*, Firma KLM, Calcutta, 1991, as Vol. 2, *Formation of the Theoretical Fundamentals of Natural Science*, Appendix IV, pp 417–432.

[21] S. C. Vidyabhushan, *A History of Indian Logic*, Calcutta, 1921. Neither Jaina nor Buddhist records tell us which Bhadrabahu was associated with *syāvāda*. I am inclined to think it was Bhadrabahu the junior, a contemporary of Dinnaga, and not the senior.

[22] C. K. Raju, *Time: Towards a Consistent Theory*, cited earlier, Chapter 6b. The differences of logic are related to differences in time perceptions, explicitly articulated, in the Buddhist case in the theory of conditioned coorigination.

mathematical truths. Logic is the key principle used to decide mathematical truth but it is not clear how this principle is to be fixed without bringing in social and cultural considerations.

We see that the "universal" reason of the schoolmen was underpinned by the alleged authority of God to which the schoolmen indirectly laid claim. If this authority is denied, as Buddhists inevitably would, there is nothing except practical and social authority that can be used to fix the logic used either within a formal theory or in metamathematics.

To summarise, all of present-day mathematics, in practice, or in principle, depends upon social and cultural authority.

1.3 Śūnya

Though the general proposition is established, an example of an alternative mathematics is still lacking. Intuitionism is an example, but not an entirely satisfactory one. Alternative mathematical theories, such as the one above which gives $2+2=5$, are not to be found very easily in history, just because historically people depended on the manifest and the empirical.

Indeed, one way of making mathematical truths independent of social and cultural authority is to change the philosophy of mathematics to admit the empirical. One must allow physics to decide the nature of mathematics. This would not damage any applications of mathematics to the empirical sciences, but mathematical truths would become contingent or of limited applicability like physical theories. An excellent example of this is the mathematics needed for extracting finite parts from the divergent integrals arising in the S-matrix expansion (renormalisation problem) of quantum field theory. All predictions of quantum field theory are based on this process[23]

Clearly, however, situations may arise were the manifest fails to provide any guidance about mathematics. This is where one can locate differences. By way of illustration consider division by zero. A more complicated case of this sort is when 0 itself is divided by 0. Brahmagupta asserted[24] that $\frac{0}{0} = 0$. Was this incorrect? In answering this question, it has been overlooked that different sorts of answers are possible, even within present-day mathematics.

(1) Look upon $\frac{0}{0} = 0$ as a limit: $\lim_{x \to a} \frac{f(x)}{g(x)}$, when $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$. The answer is provided

[23] Many physicists still believe that this mathematics is non-rigorous, but it "works". Both beliefs are false. The process of extracting a finite part from the divergent integrals of quantum field theory amounts to a definition, and one can hardly question the "rigor" of a definition. The usual definition manifestly does *not* work for it does not work even for polynomial lagrangians of degree greater than 4. It also does not work when it is applied to the same mathematical problem in classical field theory, e.g. in connection with shock waves, or singularities. In this situation one may decide between alternative mathematical theories by considerations of widest possible applicability. See, C. K. Raju, "Distributional Matter Tensors in Relativity", *Proc. MG5*, D. Blair and J. Buckingham (eds), World Scientific, Singapore, 1989, p 421–23.

[24] B. B. Dutta and A. N. Singh, *History of Hindu Mathematics, A Source Book*, Parts I and II, Asia Publishing House, Bombay [1935] 1962, p 245, "Brahmagupta has made the incorrect statement that 0/0=0."

by l'Hospital's rule. In this case, $\frac{0}{0} = 0$ can assume any real value.

(2) Look upon $\frac{0}{0} = 0$ as a convention in the context of the extended reals: $\mathbb{R} \bigcup \{\infty, -\infty\}$. Measure theory and the theory of the Lebesgue integral require the convention $0 \cdot \infty = 0$, to permit a function to be infinite on a set of measure zero, without altering the integral. It is possible that the infinity of the function in question arises because $f = \frac{g}{h}$ and $h = 0$. In this case, there is implicit acceptance of the convention that $\frac{0}{0} = 0$.

(3) Look upon $\frac{0}{0}$ as a ratio of two infinitesimals, i.e., numbers are represented by a non-Archimedean field as was implicitly done in dealing with infinitesimals in Europe before the advent of formalism. (Brahmagupta had observed that the differences in a sine table are proportional to cosines.) In the particular case that the non-Archimedean field is ${}^*\mathbb{R}$ the non-standard extension of reals , we recover the result provided by elementary calculus, viz. that $\frac{0}{0}$ may have any value.

(4) I have argued that *Śūnya* does not simply mean 0: "non-representable" is closer to its earliest usage in Indian mathematics and in *Śūnyavāda*; "non-representable" may refer to a number that is either too small or too large. It refers both to the infinitesimal and the infinite. Thus, one may look upon $\frac{0}{0}$ as a ratio of two non-representable numbers. One can distinguish two senses of the term "non-representable". In the first situation, one is working on a finite set of numbers, which is not a field, since associative and commutative laws also may fail. This is the situation for numbers represented on a modern-day computer. On current computers, for most compilers and programming

languages I am aware of, any large-enough or small enough number is non-representable as a basic data type. Non-representability of very large or very small numbers necessarily holds, in fact, for any machine with a finite memory, though it is not true of the idealised Turing machine. On actual computers, the rule that "division of a non-representable number by a non-representable number leads to a non-representable number" is applied without exception, as may be ascertained by writing a small C program[25]

(5) Finally, and I think this is closest to Brahmagupta's point of view, one could look upon $\frac{0}{0}$ as a ratio of two non-representable numbers, and use intelligent rule-and- exception in place of mechanical rule-without-exception. That is, one could use the rule most of the time, but one could cancel "non-representable" numbers in appropriate exceptional cases: e.g. $\frac{2\cdot 10^{\pm 200}}{10^{\pm 200}} = 2$, corresponding to $2 \cdot \frac{0}{0} = 2$! This is exactly how Bhaskara II interprets the rule while computing the value of x (= 44), given that[26]

$$\frac{x \cdot 0 + \frac{x \cdot 0}{2}}{0} = 63$$

We see that the answers to the question "What is $\frac{0}{0}$?" are to be decided by convention, so that no particular

[25]The full C program may be reproduced here.
#include <stdio.h>
void main (void) float a, b, c; a=1.0E39; b=-2.0E39; printf ("\n a=%E, b=%E", a, b); c=a/b; printf ("\n c=%E, c);} The output of this program would be a = +INF, b= –INF, followed by a floating point error.

[26]Datta and Singh, cited earlier, p 245. There is a misprint there about the value of *x*.

answer can be stated to be "wrong". What ought to decide the convention? The clear answer enunciated by Poincaré is: convenience.

I conclude with the hope that the above reasoning about reason demonstrates conclusively that present-day uniformity of mathematical opinion depends upon social and cultural factors that are NOT universal. Further, the above examples from both geometry and analysis demonstrate how variation in cultural factors can lead to a variation also in the content of mathematics, even at an elementary level.