Raju, C. K. 'Calculus Transmission'. In Encyclopedia of Non-Western Science, Technology and Medicine, edited by Helaine Selin, 1016–22. Dordrecht: Springer, 2016

Calculus transmission

C. K. Raju

It has gone unnoticed that Āryabhaṭa's 24 sine values (see CALCULUS) involved a striking departure from the earlier geometric tradition, and a paradigm shift to numerical techniques. The geometric tradition is useful only in simple situations where high symmetry is present. It cannot be used to calculate sine of 1 degree, but was earlier used to compute 6 sine values 15 degrees apart.

In a second radical departure, Āryabhaṭa used difference equations, instead of algebraic equations. Indeed, in the tenth *gītikā* (this is called the tenth *gītikā* since the first two are invocations; Shukla and Sarma, 1976), of the *daśgītikā* section, he states only the sine *differences*:

मखि भखि फखि धखि णखि ञखि
ङखि हस्झ स्ककि किष्ग श्घकि किघ्व ।
घ्लकि किग्र हक्य धकि किच
स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्या ॥१२॥

The numbers involved here are expressed in Āryabhaṭa's novel numerical notation (See, Shukla and Sarma, 1976, or Raju, 2007, chp. 3). Thus, the verse may be translated:

> 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7—[these are the] sine [lit. half-chord] differences [for the quadrant divided into as many equal parts, each part hence being 225′] [in] minutes.

The method of calculating these differences is explained in Gaṇita 12 as

प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितियार्धम् ।
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥१२॥

This may be translated

(12) The sine of the first arc divided by itself and diminished gives the second sine difference. That same first sine, when it divides successive sines gives the remaining [sine differences].

(Trans. by the author, based on the Hindi translation of Rai 1976, pp. 42–43; cf. Shukla and Sarma 1976, p. 51.)
That is, if the quadrant of the circle is divided into, say, 24 equal parts, $R_1,\ R_2,\ \ldots,\ R_{24}$ are the 24 corresponding sine values, $\delta_1\ (= R_1), \delta_2, \ldots,\ \delta_{24}$, are the corresponding sine differences, and $\delta_i = R_i\ -\ R_{i-1}$, for $i \geq 2$, then Āryabhaṭa's rule consists of the following two parts:

$$\delta_2 - \delta_1 = -\frac{R_1}{R_1}, \tag{1}$$

$$\delta_{n+1} - \delta_n = -\frac{R_n}{R_1}. \tag{2}$$

This cannot be understood as an algebraic equation, since that would lead to the wrong answers. More precisely, it can be regarded as an algebraic equation for calculating the second difference, but not for its proper purpose, which is to calculate sine differences (Raju 2007).

Āryabhaṭa's technique of solving those difference equations was later modified by Nīlakanṭha to make it accurate to thirds. Thus, the above interpretation is also the one given by Nīlakaṇṭha in his *Āryabhaṭīyabhāṣya* (Sastri 1930), except that Nīlakaṇṭha makes it more precise, by stating it in the form

$$\delta_n^{(2)} = -\left(\frac{R_n}{R_1}\right)(\delta_1 - \delta_2). \tag{3}$$

The difference here is that for Āryabhaṭa, working to the precision of minutes, $\delta_1 - \delta_2 = 225 - 224 = 1$, while this is no longer the case with Nīlakaṇṭha, working to the precision of thirds, who uses the above-stated values, $R_1 = [224; 50; 22]$ and $R_2 = [448; 42; 58]$, so that $\delta_2 = [223; 52; 36]$, and $\delta_1 - \delta_2 = [0; 57; 46]$.

Āryabhaṭa's method of solving differential equations by finite differences is today wrongly called "Euler's method" after Euler who studied the relevant Indian texts when he wrote an article on the Indian calendar in 1700. Further, the numerical solution of differential equations (rather than metaphysical existence and uniqueness theorems) are at the heart of all practical applications of the calculus even today.

Āryabhaṭa's process of setting up and solving differential equations involves only the simple "rule of 3" or what would geometrically be called linear interpolation. However, unlike the geometric notion, the arithmetic process easily extends recursively, and recursive application of this process naturally leads to second and higher order differences, and thence to the infinite series used by Madhava and others.

Note that Āryabhaṭa himself brought in the *second* difference. Today, we would rewrite the formula as

$$\delta_n^{(2)} \equiv \delta_{n+1} - \delta_n = -\frac{R_n}{R_1}, \tag{4}$$

corresponding to the idea that the second difference/derivative of the sine is proportional to the sine itself. But with finite differences a little more detail is necessary, and Āryabhaṭa also specifies the constant of proportionality.

Second differences were also used for interpolation, from Brahmagupta onwards, about a century later, and this use was continued subsequently by Vaṭeśvara, and Bhāskara II.

Brahmagupta's second-order interpolation formula, nowadays wrongly called "Stirling's formula", is stated as follows (*Uttarakhaṇḍakhādyaka*, II.1.4, Chatterjee 1970, vol. II, p. 177).

गतभोग्यखण्डकान्तरदलविकल वधाच्छतैर्नवभिराप्त्या ।
तद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम् ॥ ४ ॥

This has been translated (using Bhaṭṭotpala's 10th c. CE [Saka 888] commentary) as:

> Multiply the *Vikalā* by half the difference of the *Gatakhaṇḍa* and the *Bhogyakhaṇḍa* and divide the product by 900. Add the results to half the sum of the *Gatakhaṇḍa* and the *Bhogyakhaṇḍa*, if their half sum is less than the *Bhogyakhaṇḍa*; subtract, if greater. [The result in each case is the *Sphuṭabhogyakhaṇḍa* or correct "tabular" difference.]

Here, the underlying table is that calculated for *khaṇḍajyā*-s or sine differences for intervals that are spaced h apart, where it is assumed that $h = 15°$ or $900'$. The *gatakhaṇḍa* or "past difference" ($= \delta_n$) refers to the interval that has been crossed, and the *vikalā* ($= \theta$) is the amount in minutes by which it has been crossed at the point at which we want to interpolate. The *bhogyakhaṇḍa* ($= \delta_{n+1}$) is the one yet to come. Thus, the formula states:

$$\text{sphuṭabhogyakhaṇḍa} \;=\; \frac{\delta_n + \delta_{n+1}}{2} \;\pm\; \frac{\theta}{h}\;\frac{\delta_n - \delta_{n+1}}{2} \tag{5}$$

$$\sin(nh+\theta) \;-\; \sin nh \;=\; \frac{\theta}{h} \;\times\; \text{sphuṭabhogyakhaṇḍa}. \tag{6}$$

This amounts to

$$\sin(nh+\theta) \;=\; \sin nh \;+\; \frac{\theta}{h}\;\frac{\delta_n + \delta_{n+1}}{2} \;\pm\; \frac{\theta^2}{h^2}\;\frac{\delta_n - \delta_{n+1}}{2}. \tag{7}$$

Vaṭeśvara (Siddhānta, II.1.63–82) (in 904 CE) uses backward differences, and works with arcs that are only $56'\,15''$ apart, and still uses quadratic interpolation, explicitly giving the second of the above formulae, among many others (Shukla 1976, part I, p. 96, and part II, p. xlvi). This enabled him to improve Āryabhaṭa's sine values (accurate to the minute), and achieve accuracy to seconds.

Bhāskara II, while offering a justification for these formulae stated by Vaṭeśvara, brings in the present or instantaneous sine difference as the mean value of the past and future sine difference. This concept of *tātkālika bhogya khaṇḍa* or instantaneous difference leads him naturally to the concept of *tātkālika gati* or *instantaneous velocity* of a planet, since it is in that context of planetary motion that these interpolations were typically required (*Siddhānta Śiromaṇi*, *Spaṣṭādhikāra* 36–38, and accompanying autocommentary *Vāsanābhāṣya*, Sastri 1858, 1932, 1939, 1942, Sengupta 1932). This notion of derivative used for planetary velocity (*gati*) is not a "precursor" to the Newtonian notion, using fluxions, but involves an understanding of the derivative which is superior from both a practical and an epistemological perspective, as explained earlier.

The calculus in India was also used to derive accurate formulae for areas and volume—a typical application of present-day integral calculus. The volume of a sphere was first correctly expressed by Śrīdhara in his *Triśaṭikā*. (Dwivedi 1899, Ramanjachari and Kaye 1912).

Bhāskara II suggests a very interesting pedagogical demonstration, involving a model of the earth made of clay or wood.

He concludes:

> This is as I have said in my Arithmetic: (*Lilāvatī*, rule 201; Sarma 1975) the area of a circle is equal to the product of the circumference by one-fourth of the diameter [πr^2]. That result multiplied by 4 gives the surface of the sphere [$4\pi r^2$], which is like the net surrounding a hand ball; the same (surface of a sphere) when multiplied by the diameter and divided by six [$\frac{4}{3}\pi r^3$] becomes invariably the volume of the sphere.

These correct formulae for the the surface area and volume of a sphere, are significant, since the correct formulae for the volume of a sphere was not known earlier.

The term "Kerala school" used by the late K. V. Sarma, has been misused to suggest that calculus originated solely in Kerala. As we have seen, the whole technique of numerically solving differential equations began with Āryabhaṭa who did this work in Kusumpura or Patna. K. V. Sarma, himself, denied that Āryabhaṭa was somehow from Kerala (Sarma 2001). Further, as the title of Nīlakanṭha's *Āryabhaṭīyabhāṣya* says, Nīlakanṭha regarded himself as belonging to the Āryabhaṭa school. This is also significant since Nīlakanṭha, as his name "somasūtvan" says, was a Brahmin of the highest caste, while Āryabhaṭa as his name "bhaṭa" indicates, belonged to a low caste. ("Bhaṭa" meaning slave should not be confounded with "Bhaṭṭa" which is the appellation of a learned Brahmin.) The Āryabhaṭa school in Kerala clearly shows that neither the caste divide nor the regional divide was as strong then. The development initiated by Āryabhaṭa was continued by a variety of other mathematicians, from Gujarat, Ujjain and Banaras. The first infinite/indefinite series was derived by Brahmagupta, as a technique for approximating difficult fractions, in what this author has called the "fraction series expansion". (*Brāhma Sphuṭa Siddhānta* 12.57, more details in Raju 2007, chp. 3.)

Even more importantly, a key input in the summation of the series was a general formula given by Nārāyaṇa Paṇḍit of Benares in his *Gaṇita Kaumudi* (Dwivedi, 1942, p. 123) for the *vārasaṅkalitā*. (Since Nārāyaṇa is a very common name in India, some scholars have confounded Nārāyaṇa Paṇḍit with some other Nārāyaṇa in the Kerala school.). This *vārasaṅkalitā* series involving summation of partial sums, is also mentioned by Bhaskara II, who, however, does not give a general formula for summing it. Some five centuries before Mādhava, Govindasvāmin (ca. 800, *Bhāṣya* on the *Mahā Bhaṡkarīya*, iv.22; Sen 1966, p. 78) and then Udayadivākara (10th c., *Sundarī* on the

Laghu Bhāskarīya, ii.3–6; Sen 1966, p. 280), of the Āryabhaṭa school in Kerala, tried to obtain trigonometric values accurate to the thirds (i.e., third sexagesimal minute), but their values were not accurate enough; Mādhava's trigonometric values *are* however accurate to the thirds. Hence, it is clear that a key input enabling the computation to thirds is the expression for the sum of the *vārasaṅkalitā* given by Nārāyaṇa Paṇḍit of Benares in his *Gaṇita Kaumudi* Thus, there is no way that the calculus developments can be localised to any one region or caste in India.

What was the social and practical need for such highly accurate sine values? This is related to the two key sources of wealth in India, namely agriculture and overseas trade. (For a summary account, see GJH 2007.) Since agriculture in India was (and still is) mostly monsoon driven, it required a good calendar to get the right time of the monsoons, described in the Indian luni-solar calendar by the months of *sawan* and *bhadon*. There is no such possibility in the Gregorian calendar. The Indian calendar was based on a sophisticated astronomical model, similar, but not identical, to the one attributed to the supposed Greek author of the *Almagest* (Raju 2014).

In this context, some authors refer to the sine as Rsine, and claim that trigonometry was transmitted to Indians from the *Almagest*. However, the actual evidence tells us that trigonometry was transmitted from India to Europe via Arabs. Indeed, the very word "sine" derives from a translation howler during the Toledo mass translations of Arabic texts in 1125 CE. The word derives from the Latin *sinus* or fold, which is a translation of the Arabic *jaib*, meaning pocket, which is a wrong reading of the Arabic *jiba*, from the vernacular *jīvā*, for the Sanskrit *jyā*, meaning chord (as used by Āryabhaṭa).

Indeed, scientific texts are accretive, so that the the 12th c. *Almagest* reflects 12th c. knowledge which came about by transmission from India. There is no serious evidence for either its existence in its present form in the 2nd c. CE, or that its author was a Roman-Greek called Claudius Ptolemy. The failure to reform the Julian calendar since the 5th c. is counter evidence against this belief (Raju, 2010).

The linguistic error of translation in the term "sine" was accompanied by a conceptual error, as in the very word "trigonometry" where the functions relate to the circle, not the triangle. That error persists to this day in the teaching of "trigonometry" which is stuck in the pre-Āryabhaṭa era. The word "trigonometry" is in quotes, since this geometric method wrongly suggests that the concepts of sine and cosine relate to the triangle, whereas they actually relate to the circle.

Studying the movements of planets on the celestial sphere naturally required calculations with the circular functions sine and cosine. Hence, sine tables are commonly found in Indian astronomy and mathematics texts, starting from the 3rd c. CE *Sūrya Siddhānta*. This requirement of an accurate calendar for successful agriculture still persists. (Specifically on a number of occasions in the last decade there has been large-scale crop failure due to mistimed agricultural operations; Raju 2007, chp. 4.)

Accurate sine values (and an accurate calendar) were also required for navigation even to determine latitude. At night, latitude could be determined simply by measuring the altitude of the pole star. This was done using a sophisticated instrument (with harmonic interpolation) which was never earlier understood by Western historians (Raju 2007, chp. 5). This instrument was used by the Indian navigator who brought Vasco da Gama to Calicut in India from Melinde in Africa. However, Vasco neither knew nor immediately understood the method. Since the Malayalam word for the pole star (*kau*) also means teeth, and the instrument used by the navigator is held between the teeth, Vasco da Gama recorded that "the pilot told the distance by his teeth"!

In day time, latitude was determined from observations of solar altitude at noon, and longitude was determined by solving the appropriate longitude triangles (Raju 2007, chp. 4). A solution to these two problems of latitude and longitude determination is found in many early Indian texts, such as the text of the 7th c. Bhaskara 1, a commentator on the 5th c. CE Āryabhaṭa.

The 7th c. Bhaskara-1 explains (*Laghu Bhāskarīya*, III.2-3, Shukla 1963, p. 42, Raju 2007, chp. 4) how to calculate the local latitude (ϕ) from the equinoctial midday shadow (s) using a gnomon of height g using the formula $\tan\phi = \frac{s}{g}$. Note that this formula is useful only if one has a method of computing the arctangent function (or a method of "arcifying" sines), and this explains the need for the infinite series for the arctangent.

The 7th c. Bhaskara-1 also explains (*Laghu Bhāskarīya*, III.22–23) how to calculate the local latitude from observation of solar altitude at noon, provided also that the solar declination is known. That requires a good calendar which correctly tells the date of the equinox.

Bhaskara-1 also gives a method of longitude determination, by solving a longitude triangle, from a knowledge of the departure or distance between two points A and B, and the corresponding difference of latitude. (*Mahā Bhāskarīya* II.3–4, Shukla 1960, pp. 49–50). He himself criticised the rule, among other reasons, for neglecting the sphericity of the earth, and not using spherical triangles.

This solution of the longitude problem required an accurate knowledge of the size of the earth. Columbus underestimated the size of the earth by about 40%, and the resulting navigational disasters led to the Portuguese ban of 1500 on carrying globes aboard ships. In contrast, most Indian mathematics and astronomy texts also state an estimate of the radius of the earth. This is stated in *yojanas*, and we don't know *exactly* how much a yojana was. Nevertheless, we do know that al Biruni came to India, studied astronomy, and reported on Indian astronomy in his 10th c. *Kitab al Hind*. His estimate was stated in Arabic miles, and intended to corroborate Khalifa al Mamun's direct measurement of one degree of arc in the desert. This method of calculating the size of the earth (Raju 2007, chp. 4) required very accurate sine and cosine values, and it is also the method implicit in the definition of the the Arabic *zam* (from the Sanskrit *yāma*), as the "distance from here to the horizon" (Raju 2007, chp. 5). Al Biruni's error was 0.25%, compared to the error of 40% in Columbus' estimate, and the error of 25% in Newton's first estimate (Rizvi 1979).

Further, in contrast to Indo-Arabic techniques of celestial navigation, the European technique of navigation involved charts. Hence, apart from the issue of latitude and longitude at sea, a key problem of European navigation was that of loxodromes: because a chart is flat while the surface of the earth is curved, holding a straight course in one direction (using a magnetic compass or a straight line joining two stars) does not result in a straight line on the surface of the earth, but a curved line, called a loxodrome. The solution to this problem of loxodromes was provided by the Mercator chart which used an old projection technique of Chinese (Dunhuang) star charts (Needham 1981, pp. 123-124) to map loxodromes to straight lines. Constructing the Mercator chart required an accurate table of secants, or, equivalently, accurate sine values. (The formulae for the Mercator projection are widely available; see, e.g., Raju 2007, p. 339.) An accurate technique of navigation was the biggest scientific challenge in Europe then. Hence, the Jesuits in Cochin were naturally interested in these Indian texts with accurate sine values. Their college in Cochin helped procure and translate local Indian texts, and sent them back to Europe in a repeat of the Toledo model of mass translation of books. In place of the Mozharabs of Toledo, they then had the support of the local Syrian Christian community in the vicinity of Cochin.

The translated Indian texts would naturally have gone first to the Jesuit general. There is ample circumstantial evidence that did happen. Christoph Clavius, who authored the Gregorian calendar reform, also published in his name a table of sines in 1607. Curiously, these were the so-called Rsines, in that they explicitly involved the radius of the circle. Simon Stevin follows the same practice for his secant tables. Curiously, Clavius used the same large number for the radius as used in Madhava's values (Clavius 1607). Documentary evidence of a connection comes from Clavius' student Matteo Ricci who visited Cochin just prior to the Gregorian reform to get information about Indian methods of timekeeping (Ricci 1581). The Indian timekeeping or astronomy texts near Cochin contained detailed accounts of the calculus.

On the epistemic test, those who copy don't fully understand what they copy. This is also evidence of transmission: Clavius got the imported sine values explicitly interpolated to build a larger table, but did not know enough trigonometry to calculate the size of the earth. Recall that this size was routinely mentioned in Indian texts, and that the size of the earth was a key parameter needed for determining longitudes. (The calendar reform only settled the problem of latitudes.)

There is other circumstantial evidence of transmission of calculus to Europe. Clavius' contemporary, Julius Scaliger, is credited with Julian day-number system which is the same as the Indian *ahargaṇa*. Likewise, another contemporary Tycho Brahe, Royal Astronomer to the Holy Roman Empire, produced the Tychonic astronomical model (in which all planets go round the Sun, which itself goes round the earth) which is just a carbon copy of the astronomical model of Nīlakanṭha, stated in his *Tantrasangraha*. Tycho's masonry instruments (copied from Ulugh Beg's Samarkand observatory) were not accurate enough to make accurate observations of Mars, such as made by Parameswaran over a 50 year period. Nevertheless, Tycho, in those days of the Inquisition, kept some secret documents with which his assistant Kepler decamped, after Tycho's untimely death or murder. Why did Tycho keep mere observations such a secret from his own assistant? How did Kepler, a nearly blind person, arrive at those super-accurate observations, without appropriate instruments? (Donahue 1988, Broad 1990.)

Likewise, Fermat's challenge problem to European mathematicians, which remained unsolved for long (and was eventually solved by Euler) is taken from an explicitly solved example in Bhaskara's *Bījagaṇita* (87, Colebrooke 1816, pp. 176–178). Indeed, Bhaskara himself poses it as a challenge, saying "Declare it friend if the method [of solution] be spread over your mind like a creeper".) Thus, in Feb 1657, Fermat (*Ouvres*, p. 332 et seq.) asked European mathematicians to solve the problem $Nx^2+1 = y^2$ for a given (positive, non-square) N. As examples, he listed, for the case $N = 3$, that $x = 1, y = 2$ are solutions, and $x = 4, y = 7$ are also solutions. Then he asked for the smallest integer solutions for the case $N = 61$, and $N = 109$. This is today called "Pell's equation", and the smallest solutions are the numbers $x = 226153980$, $y = 1766319049$ given by Bhaskara II centuries earlier. Given how large these numbers are, an independent rediscovery would represent a fantastic coincidence.

When the Indian calculus first reached Europe, some people like Fermat and Pascal accepted it enthusiastically, while other people like Liebniz did not fully understand it. (Newton, in his anonymous review of his own report, on behalf of the Royal Society, on Leibniz's charge of plagiarism against him, claimed that Leibniz failed to comprehend the infinite series named after him; Newton 1714.) Descartes too was unable to comprehend how to sum infinite series. Indeed, Descartes opined that the ratio of curved and straight lines was beyond the capacity of the human mind (Descartes, 1996). In India children were taught to measure angles as the length of a curved line (the arc) using a flexible string since the days of the śulba sūtra-s (or aphorisms on the string) (Raju 2009). They could easily straighten the string to compare the length of a curved line with a straight line (and needed to do so to measure the arc in units of the radius).

On the epistemic test, mistakes are proof of transmission. A charitable interpretation of Descartes' blunder is that he was presumably alluding to the infinite series for π, found in Indian texts from long before Leibniz. He thought an infinite series could only be summed by actually carrying out an infinite number of sums, and hence regarded an infinite sum as a supertask. Descartes was not the only person who thought thus. Galileo, whose access to the Jesuit archives is well documented, in his letters to Cavalieri (Mancosu 1966) agreed with this objection, hence eventually left it to his student Cavalieri to take the credit or discredit for calculus.

Newton, himself, claimed credit only for making the calculus rigorous, not for inventing it (though he did claim credit for the sine series). He thought the calculus could be made rigorous by making time metaphysical. Ironically, that was the precise reason why his physics failed (Raju 1994). Changing that understanding of time/calculus improves physics including the theory of gravitation (Raju 2012).

Bibliography

- Broad, W. J., 1990, "After 400 Years, a Challenge to Kepler: He Fabricated Data, Scholars Say", *New York Times*, 23 January 1990, C1, 6. Also, 'Planet Fakery Exposed. Falsified Data: Johannes Kepler". *The Times* (London) 25 January 1990, 31a.

- Chatterjee, B. (ed. and trans.) 1970, Brahmagupta, *Khaṇḍakhādyaka*, with commentary of Bhaṭṭotpala, Motilal Banarsidass, New Delhi.

- Colebrooke, H. T. (trans.), 1816, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara*, London, John Murray.

- Clavii, Christophori Bambergensis, 1607, *Tabulae Sinuum, Tangentium et Secantium ad partes radij 10,000,000...*, Ioannis Albini.

- Descartes, R., 1996, *The Geometry*, trans. David Eugene and Marcia L. Latham, Encyclopedia Britannica, Chicago, Book 2, p. 544.

- Donahue, W., 1988, "Kepler's Fabricated Figures: Covering Up the Mess in the New Astronomy", *Journal for the History of Astronomy*, **19**, pp. 217–37.

- Dwivedi, Sudhakara (trans.) 1899, Śrīdhara, *Triśaṭikā*, 56, Benaras, trans. N. Ramanaujachari and G. R. Kaye, *Bibliotheca Mathematica*, **13** (1912) pp. 203–17.

- Dwivedi, Padmakara (ed.) 1942, Nārāyaṇa, *Gaṇita Kaumudi*, Princess of Wales Sarasvati Bhavana Texts, No. 57, Benares, part 1.

- GJH 2007, "Cultural Foundations of Mathematics", *Ghadar Jari Hai*, *2*(1) pp. 26-29, available online at `http://IndianCalculus.info/Cultural-Foundations-Mathematics-review-GJH.pdf`

- Mancosu, P., 1996, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, Oxford University Press.

- Needham, J., 1981, *The shorter science and civilisation in China*, vol. 2, abridged by C. A. Ronan, Cambridge university press, 1981.

- (Newton, I.) 1714, "An account of the book entituled Commercium epistolicum collinii Et aliorum, de analysi promota", *Philosophical Transaction of the Royal Society of London*, No. 342. Jan. and Feb. 1714/15, pp. 173–224.

- Rai, R., 1976, *Āryabhaṭīya*, INSA, New Delhi.

- Raju, C. K., 1994, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht.

- Raju, C. K., 2006, "Time: what is it that it can be measured?" *Science & Education*, **15**(6), pp. 537–551.

- Raju, C. K., 2007, *Cultural Foundations of Mathematics*, Pearson Longman, Delhi.

- Raju, C. K., 2009 "Towards Equity in Math Education 2. The Indian Rope Trick", *Bharatiya Samajik Chintan*, New Series, **7** (4), pp. 265–269.

- Raju, C. K., 2010, *Is Science Western in Origin?*, Multiversity and Citizens International, Penang, 2010.

- Raju, C. K., 2012, "Retarded gravitation theory" in: Waldyr Rodrigues Jr, Richard Kerner, Gentil O. Pires, and Carlos Pinheiro (ed.), *Sixth International School on Field Theory and Gravitation*, American Institute of Physics, New York, pp. 260–276. `http://ckraju.net/papers/retarded_gravitation_theory-rio.pdf`.

- Raju, C. K., 2014, "A tale of two calendars", in *Multicultural Knowledge and the University*, ed. Claude Alvares, Multiversity and Citizens International, Penang, chp. 10, pp. 112–119.

- Ricci, Matteo, 1581, Letter to Petri Maffei on 1 Dec 1581. *Goa* **38** I, ff. 129r–30v, corrected and reproduced in *Documenta Indica*, **XII**, 472–477 (p. 474). Also reproduced in Tacchi Venturi, *Matteo Ricci S.I., Le Lettre Dalla Cina 1580–1610*, vol. 2, Macareta, 1613.

- Rizvi, S. S. H., 1979, "A newly discovered book of Al-Bīrūnī: 'Ghurrat-uz-Zijat', and Al-Bīrūnī's 'Measurements of Earths Dimensions' ". In: *Al-Bīrūnī Commemorative Volume*, ed. H. M. Said, Hamdard Academy, Karachi, pp. 605–80.

- Sarma, K. V. (trans.), 1975, *Līlāvatī of Bhāskarācārya with Kriyākramkarī* cr. ed. by VVBI, Hoshiarpur.

- Sarma, K. V., 2001, "Āryabhaṭa: his name, time and provenance", *Indian J. History of Science*, **36** (3–4) pp. 105-115.

- Sastri, Bapudeva, 1858, "Bhāskara's knowledge of the differential calculus", *Journal of the Asiatic Society of Bengal*, **27** pp. 213–16.

- Sastri, Bapudeva (ed.) 1929, *Siddhānta Śiromaṇi*, rev. by Ganapati Deva Sastri, Benares, (Kashi Sanskrit Series, No. 72)

- Sastri, Bapudev (ed.) 1939, *GrahaGaṇitadhyāya* with *Vāsanābhāṣya*, . Anandashram Sanskrit Series, 110, Poona, pp. 52–53.

- Sastri, Bapudev (ed.) 1943, *Golādhyāya* with *Vāsanābhāṣya* and com. Marici of Munisvara, Anandashram Sanskrit Series, 122, Poona.

- Sastri, K. S. (trans.), 1930, *Āryabhaṭīya of Āryabhaṭācārya with the Bhāṣya of Nīlakaṇṭhasomasūtvan*, ed. University of Kerala, Trivandrum, reprint 1970.

- Sen, S. N., 1966, *A Bibliography of the Sanskrit works on Astronomy and Mathematics*, INSA, New Delhi.

- Sengupta, P. C., 1932, "Infinitesimal calculus in Indian mathematics", *Journal of the Department of Letters* (Calcutta University) **22** pp. 1–17.

- Shukla, K. S. (trans.), 1960, *Mahā Bhāskarīya*, Lucknow University, Lucknow.

- Shukla, K. S. (ed. and trans.), 1963, *Laghu Bhāskarīya*, Department of Mathematics and Astronomy, Lucknow.

- Shukla, K. S.(ed. and trans.) 1976, *Vaṭeśvara Siddhānta Gola of Vaṭeśvara*, Indian National Science Academy, New Delhi.

- Shukla, K. S. and Sarma, K. V. (trans) 1976, *Āryabhaṭīya of Āryabhaṭa*, INSA, New Delhi.