Calculus Without Limits

C. K. Raju

Indian Institute of Education
G. D. Parikh Centre, J. P. Naik Bhavan
Mumbai University Kalina Campus
Vidyanagari, Santacruz East, Mumbai 400 098
ckr@ckraju.net

Abstract

The course aims to teach calculus as it developed in India, since the 5th c. Aryabhata, as the numerical solution of differential equations. Instead of formal real numbers and limits, this uses the *avyakta* numbers of Brahmagupta (today known as the non-Archimedean field of rational functions), combined with the philosophy of zeroism which involves discarding or zeroing of infinitesimals and small quantities (for example in summing infinite series, as in the sum of an infinite geometric series given by Nilakantha). This Indian method is ideally suited for implementation on present-day high-speed computers using floating-point numbers, and my software CALCODE would be used for this course.

Compared to usual university calculus courses, this course offers several advantages. (1) It leads to greater conceptual clarity (compared to formal real numbers and limits which students and even teachers rarely understand correctly). (2) It is far easier than the usual dreaded calculus courses. (3) It hence enables students to solve much harder problems never covered in usual calculus courses, such as non-elementary elliptic integrals. (4) Unlike usual calculus course, real, practical applications to physics and engineering are covered as actually done today. (5) The usual calculus course emphasizes the "skill" of symbolic manipulation (tricks to calculate derivatives and elementary integrals) but those are useless today, given existing open-sources software for symbolic manipulation such as MAXIMA which would be taught as part of the course.

Course contents

Brief historical background. How and why Europeans misunderstood Indian arithmetic: Gerbert and apices, Fibonacci and Florentine law against zero. The terms "zero" and "surd". When Europeans first learnt fractions. Misunderstanding of trigonometry: the term "sine". Conceptual confusion: trigonometric functions or circular functions; what is sin (120^o)? Measurement of angles through arc: radians. Corresponding definition of sine as a circular function.

Geometry vs numerics: Present-day teaching of 5 trigonometric values vs Aryabhata's and Madhavs's 24 sine values: progress or regress? Geometry works only in cases where there is symmetry, numerics works everywhere. Why sine differences (*khanda-jya*) not sine values? Sine differences and the elementary rule of three (cross-multiplication) as linear interpolation. Calculating sin (1^o) in various ways. Finite differences vs derivatives: chord vs tangent. What is the exact definition of tangent? Error and myth of perfection. Zeroism.

Differential equations basics: Relation between values and differences: the "fundamental theorem of calculus". Observation that sine differences are proportional to cosine. Observation about second sine differences. Differential equations vs difference equations. Solving differential equations to calculate sine values. Extending this technique: solving differential equations for the exponential function. Using

CALCODE. Constructing your own solver. Aryabhata-Euler method.

Applications of ordinary differential equations: All problems of Newtonian physics involve the solution of ordinary differential equations. Example problems: the amplitude dependence of the time period of the simple pendulum. Jacobian elliptic functions sn, cn, and dn. At what angle should one throw a javelin? A cricket ball? A tennis ball? Solving the 2-body problem of Newtonian gravitation. Example of calculating the trajectory of real spacecraft using NASA Horizons interface. Examples of chaotic motion: the Brusselator as chemical clock, and the Lorentz model. Various other problems.

Symbolic manipulation. Introducing MAXIMA. Using MAXIMA for high precision arithmetic calculations. Using MAXIMA for symbolic manipulation and calculating symbolic derivatives and integrals. Elliptic integrals.

Number systems and limits: The origin of formal real numbers: the “error” in the first proposition of the *Elements*. Dedekind cuts. The problems of Cantorian and naive set theory. Russell's paradox. Limits and Cauchy sequences. Formal reals as the largest Archimedean ordered field. Rational functions as an example of non-Archimedean ordered field. Rational functions and Brahmagupta's *avyakta* ganita. Non-existence of limits in a non-Archimedean ordered field. Obtaining limits by discarding infinitesimals, example of infinite geometric series. Relation to limits by order-counting. Computers and floating point numbers. Floats do not constitute a field: failure of associative law for addition. Extended precision arithmetic: limitation of finite memory, infinite precision arithmetic is not possible in finite time. The problem with ints on a computer.

Higher order polynomial interpolation. Brahmagupta, Vateshwar-Stirling formula and quadratic interpolation. Runge-Kutta methods and higher order polynomial interpolation. Madhava-Taylor series. The Madhava-Leibniz series for π. Its rate of convergence. Accelerating convergence.

Apart from classroom lectures, the course will require the use of laptops desktops by the students.

Earlier teaching of decolonised calculus courses: some examples

Calculus Without Limits

Do you know your limits?

Did you know calculus originated in India? Europeans stole it in the 16th c. but failed to understand how Indians summed infinite series (without limits). Watch the videos of C. K. Raju's lectures on calculus at MIT (`https://youtu.be/IaodCGDjqzs`), and Indian Institute of Science (`https://youtu.be/U-r1CWU-KKM`). The original calculus is easier and better. It is well adapted to computers. You can use it to solve harder problems today.

- Targeted to students of engineering and science
- Course will run for only 20 hours.
- Course starts on 24 August 2017.

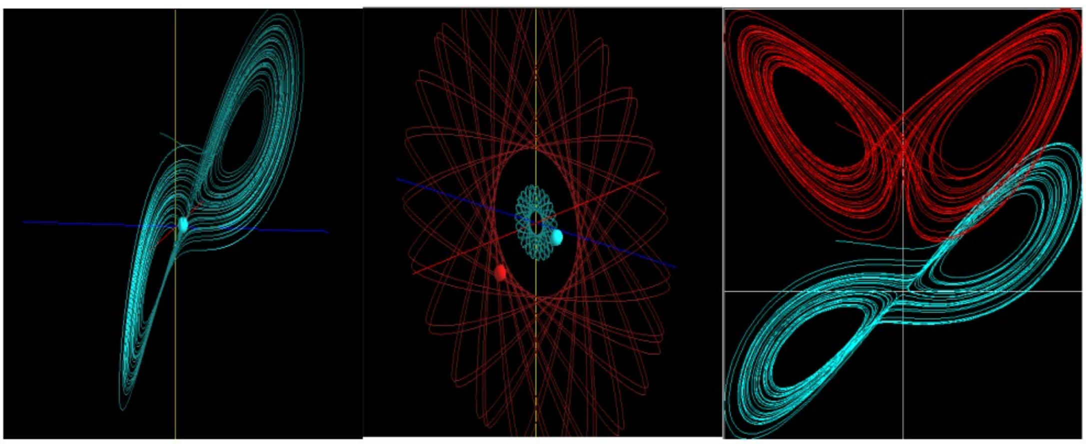
What will you learn?

- Basic concepts of calculus on a new philosophy of zeroism.
- Calculus as the *gaṇita* of differential equations.
- Also, how to do symbolic manipulations using a computer.

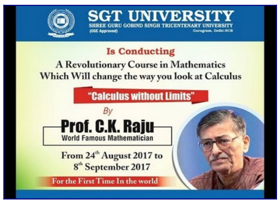
Outcomes

At the end of the course you should be able to do the following math problems in a jiffy (using a computer):

- Evaluate most derivatives and integrals.
- Numerically solve ordinary differential equations, and analyse the solutions.
- Apply this knowledge to build realistic mathematical models.



Poster SGT University Delhi NCR (science and engineering undergrads) (above)

Video of initial lecture: 

Group photo:

2. Ambedkar University Delhi

Bharat Ratna Dr B.R.
Ambedkar University, Delhi (AUD)
(Established by the Government of the National Capital Territory of Delhi)

Calculus for Social Scientists

Frightened of Mathematics?

Did not do mathematics in high school? Found it difficult? Here is an opportunity to get rid of this fear! As part of a project to make mathematics easy, AUD is offering an experimental new workshop-course by Professor C.K. Raju, Universiti Sains Malaysia, on learning to do *Calculus without Limits*.

- This workshop is designed for students and faculty from social sciences, arts, communications, education, humanities, languages, and management among others.
- Course will run for only 5 days.
- Certificate of proficiency will be provided on successful completion of the course.
- Hurry! Limited seats.
- For registration, contact Ms Manasi Thapliyal, School of Educational Studies, AUD. email: manasi@aud.ac.in.
- Course duration: **Monday, 7 May to Friday, 11 May 2012**; Venue: Computer Lab, AUD Kashmere Gate Campus, Lothian Road, Kashmere Gate, Delhi-110006.

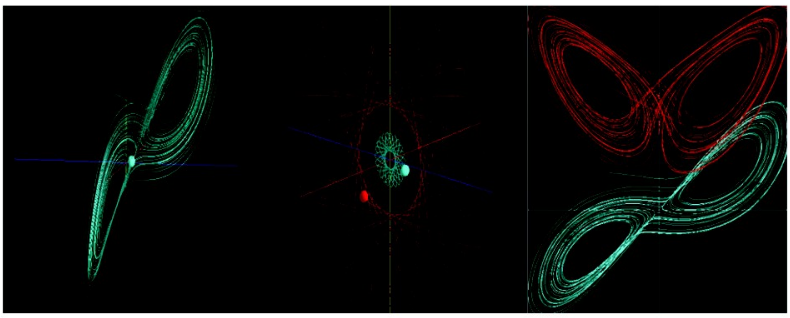
What will you learn?

- Basic concepts of calculus based on a new philosophy.
- How to do symbolic manipulations and numerical calculations using a computer. (Free software would be provided.)

Outcomes

At the end of the course you should be able to do the following math problems in a jiffy (using a computer):

- Evaluate most derivatives and integrals.
- Numerically solve ordinary differential equations, and analyse the solutions.
- Apply this knowledge to mathematical models, mathematics teaching etc.



AUD Group photo

3. CISSC, Iran
English Poster

Group photo

4. Universiti Sains Malaysia, Penang

The course was taught to four groups of students. 1 post-graduate in math, and 3 undergraduate groups: in applied math, pure math, and non-math. E. g., below of PG PRE-test to bring out serious flaws in teaching of calculus and probability which create difficulties at later level. (Easier pre-test for undergrads; assumes only school calculus.)

Math Group: Calculus without Limits
Exam, Pre-test: A

Name: ______________________
Student Number: ______________________
Course: ______________________
Age: ______________________
Date: ______________________

- Please attach this question paper and return it along with your answer sheet.
- This is not a competitive test. The aim is to obtain feedback to decide what to teach and how.
- Since the group is heterogeneous, you may find some questions too easy, or some may be too difficult. Attempt as many questions are you are able to.

1. (a) Define a complete metric space.
   (b) The least upper bound property for $\mathbb{R}$ says that if $A \subset \mathbb{R}$ is non-empty and bounded above, then $\exists \alpha \in \mathbb{R}$ such that $a \leq \alpha$, $\forall a \in A$, and if $a \leq b$, $\forall a \in A$ then $\alpha \leq b$. Assume the least upper bound property and prove that $\mathbb{R}$ is a complete metric space.
2. (a) Define "infinite set", "countable set", "uncountable set".
   (b) Prove that $\mathbb{R}$ is uncountable.
   (c) If $\mathbb{N}$ is the set of natural numbers, and $P(\mathbb{N})$ is its power set, does there exist a bijective map $f : P(\mathbb{N}) \to \mathbb{R}$?
3. (a) Write down the binary representation of 41.
   (b) Write down the binary representation of 2.5
   (c) Rewrite your answer using a mantissa between 1 and 2.
4. Given $g(x) = \begin{cases} x^2 - C, & \text{if } x < 4 \\ -\sqrt{C}x + 20, & \text{if } x \geq 4 \end{cases}$
   (a) Find the value of C which makes g continuous on $(-\infty, \infty)$.
   (b) With the above value of C, is g differentiable? Explain your answer.
5. (a) Suppose f_n is a sequence of Riemann integrable functions which converges to the function f on $(0, \infty)$, convergence being uniform on compact subsets. Is it true that f is Riemann integrable and that $\int_0^\infty f_n \to \int_0^\infty f$?
   (b) Suppose f_n is a sequence of differentiable functions which converges uniformly to the function f on $(0, 1)$. Is it true that f is differentiable and that the sequence of derivatives $f_n{}' \to f'$?

1

6. (a) Give an example of a real-valued function f which is not Riemann integrable on $[0, 1]$. Is this Lebesgue integrable?
   (b) Does there exist a Riemann integrable function on $(0, \infty)$ which is not Lebesgue integrable?
7. The following ten numbers were drawn at random from $[0, 1]$ using a uniform probability distribution: 0.23, 0.74, 0.18, 0.79, 0.51, 0.34, 0.67, 0.44, 0.11, 0.44.
   (a) Find the average.
   (b) Explain why it is not 0.5.
   (c) If the average does equal 0.5 at some stage, can subsequent draws of further random numbers change that value?
   (d) An unbiased coin is tossed 100 times. The first toss is tails, and the subsequent 99 tosses are heads. At the 101st toss (i) is the probability of tails greater than that of heads or (ii) is the probability of heads greater than that of tails?
8. Suppose a monkey is typing at random on a typewriter which has 50 keys (z and Z having been dropped), and suppose that the monkey is equally likely to strike any key.
   (a) What is the probability that the first six letters the monkey types will spell the word "Hamlet".
   (b) Suppose we consider the letters typed by the monkey in consecutive blocks of six letters. What is the probability p_n that the first n blocks of six letters will have the word "Hamlet"?
   (c) Does $\lim_{n\to\infty} p_n$ exist? If so, what is it?
9. Differentiate the following with respect to x
   (a) $\sin^n x . \sin nx$
   (b) $\sec^{-1} \frac{\sqrt{x}+1}{\sqrt{x}-1} + \sin^{-1} \frac{\sqrt{x}-1}{\sqrt{x}+1}$
   (c) $x - \log\left(2e^x + 1 + \sqrt{e^{2x} + 4e^x + 1}\right)$
10. Evaluate the following indefinite integrals.
   (a) $\int \sqrt{3x+2}\, dx$
   (b) $\int \log x \, dx$
   (c) $\int \frac{dx}{\sqrt{\sin^3 x \cdot \sin(x+\alpha)}}$

2

No group photo.

Report:
C. K. Raju, 'Teaching Mathematics with a Different Philosophy. 1: Formal Mathematics as Biased Metaphysics', *Science and Culture* 77, no. 7–8 (2011): 274–79; arXiv:1312.2099;
C. K. Raju, 'Teaching Mathematics with a Different Philosophy. 2: Calculus without Limits', *Science and Culture* 7, no. 7–8 (2011): 280–85. arXiv:1312.2100 (3 undergrad groups, and one post-graduate math group.)

5. Central University of Tibetan Studies, Sarnath, Varanasi, 2009.

Group photo.

Report: Raju, C. K. ‘Calculus without Limits: Report of an Experiment’. In *2nd People’s Education Congress*. TIFR, Mumbai: Homi Bhabha Centre for Science Education, 2009.
Presentation: http://ckraju.net/papers/Calculus-without-limits-presentation.pdf.

Also, same title, published in proceedings http://ckraju.net/papers/calculus-without-limits-paper-2pce.pdf.

6. Original Indian philosophy of math and calculus explained in more detail, at popular level, in two recent articles.

Raju, C. K. ‘California, Indian Calculus and the Technology Race. 1: The Indian Origin of Calculus and Its Transmission to Europe’. *Boloji.Com*, 11 December 2021. https://www.boloji.com/articles/52924/california-indian-calculus.
———. ‘California, Indian Calculus and the Technology Race. 2: Don’t Cancel the Calculus, Make It Easy!’ *Boloji.Com*, 24 December 2021. https://www.boloji.com/articles/52950/california-indian-calculus-and.

Tutorial-1
Calculus without Limits

1. (a) Define an angle
   (b) Convert 32°into radians.
   (c) Convert 0.78 radians to degrees.

2. (a) Solve the ODE $y' = y$ with $y(0) = 1$.
   (b) Hence, calculate the value of e.
   (c) Define the exponential function e^x.

3. (a) Convert the second order ODE $y'' = -y$ to two first order ODEs.
   (b) Solve the system of two simultaneous ODEs with the initial data $y(0) = 0$, $y'(0) = 1$.
   (c) Calculate π correct to 4 decimal places.

4. (a) Define the function $\cos(x)$.
   (b) Calculate $\cos(42°)$.

5. (a) The equation for damped harmonic motion is often written as

$$\ddot{y} = -k^2 y - b\dot{y}$$

   . Convert this to a system of 2 ODEs, and solve with the initial data $y(0) = 0$, and $k = 1$, and $b = 0.1$.
   (b) How does the solution change if we use the initial data $y(0) = 1$?
   (c) Re-calculate the solution for $b = 0.2$, $b = 0.3$. Can you guess the solution for a general b?

6. The equation of motion for a simple pendulum is

$$y'^2 = (1 - y^2)(1 - k^2 y^2). \tag{1}$$

   The substitutions

$$\begin{aligned} y &= \mathrm{sn}(x) = y_1 & (2) \\ 1 - y^2 &= \mathrm{cn}^2(x) = y_2^2 & (3) \\ 1 - k^2 y^2 &= \mathrm{dn}^2(x) = y_3^2. & (4) \end{aligned}$$

converts this to 3-equations in Jacobi's form

$$y_1' = y_2 y_3, \tag{5}$$

$$y_2' = -y_3 y_1, \tag{6}$$

$$y_3' = -k^2 y_1 y_2, \tag{7}$$

Solve the above equations with the initial data $y_1(0) = 0$, $y_2(0) = 1$, $y_3(0) = 1$, and parameter k=0.4.
(b) Compare the Jacobian elliptic function $\text{sn}(x)$ with $\sin(x)$.
(c) The time period of the simple pendulum is the first zero of $\text{sn}(x)$. Calculate it.

7. (a) Van der Pol's equation is

$$y'' + \epsilon(y^2 - 1)y' + y = 0, \tag{8}$$

Convert this equation to two first order ODEs.
(b) Solve the resulting ODEs for $y(0) = 2$, and $y'(0) = 0$, and parameter value $\epsilon = 1$

8. (a) Solve the system of equations for the Lorenz model

$$y_1' = -\sigma y_1 + \sigma y_2, \tag{9}$$

$$y_2' = -y_1 y_3 + r y_1 - y_2, \tag{10}$$

$$y_3' = y_1 y_2 - b y_3. \tag{11}$$

for the parameters $b = \frac{8}{3}$, $\sigma = 10$, $r = 28$, and initial data $y_1 = 8$, $y_2 = -8$, $y_3 = 27$, over the range [-2, 2].
(b) Draw the resulting phase plots.
(c) Switch to 3-d view, and animate.

9. A ball is thrown upwards at an angle θ from a height of 10 meters. Assuming a simple model of air resistance proportional to velocity, and assuming its coordinates at any instant are (y_1, y_2), the equations of motion are given by

$$y_1' = y_3, \tag{12}$$

$$y_2' = y_4, \tag{13}$$

$$y_3' = -\frac{b}{m} y_3, \tag{14}$$

$$y_4' = -g - \frac{b}{m} y4. \tag{15}$$

where b is the drag coefficient and m is the mass of the ball.
(a) The mass of a cricket ball is 155.9 gram and the mass of a tennis ball is 58.5 gram. Assume $b = 0.01$. Both balls are thrown with the same velocity 10 m/s, at an angle of 45°. Which ball will travel further? By how much?
(b) If the angle of throw is changed to 44° (for either ball) will it travel a larger or a smaller distance?