

CHAPTER 7

India's Interaction with China, Central Asia, and West Asia in the Field of Mathematics and Astronomy

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1. INTRODUCTION

The region involved is geographically so vast and linguistically so diverse that I can not but start by confessing my lack of linguistic abilities to handle even a small part of the source material needed for a comprehensive review. Hence, I will not attempt a review, but will try to focus instead on the little new material that I will add.

Since I nevertheless have had to rely primarily on secondary sources, I must delineate some of those premises that I do not share with those sources. This delineation will also serve to elaborate the general model of information exchange that I feel is appropriate to such a study. Information transmission and information sharing are virtually synonymous with cultural interactions, so some conceptual framework is needed for information exchange. But despite the great interest in transmissions and diffusions, there is no explicitly stated theory of how information is exchanged between cultures in contact; and I do not find adequate the standard model of information transmission that is implicitly assumed in most histories of mathematics and astronomy—this includes those non-Western histories of mathematics that have simply tried to reverse the direction of information flows, wherever possible, without seriously challenging the premises of the underlying model of information exchange. So I will begin by pulling out into the open the premises of this underlying model. I will also indicate alternative principles of information sharing and information flow that will help us to understand, in a different way, what 'interaction' means.

2. MODELS OF INFORMATION TRANSMISSION

2.1. The colonial model of the history of science

There is a belief that the colonial model of history died of embarrassment when its naïveté

and designs were exposed. But perhaps it was only hibernating while it renewed its thick skin, for it has returned to participate in the civilizational clashes proposed by Huntington¹ in his attempt to initiate cultural globalization, a la Toynbee. An example from a recent history of astronomy is provided by North² who, despite Bernal,³ is still very keen to trace the source of all information flows back to a Greek fount—mathematics from Euclid and Archimedes, and astronomy from Ptolemy—and to dismiss everything else as mindless meandering or, at best, a matter of secondary importance.

Such tendencies were, of course, ubiquitous in earlier historical works. Following the Egyptians in their mathematics of astronomy, as he did in their chronology and stellar observations, Ptolemy in his text used only the chord; he did not use any present-day trigonometric function.⁴ This did not deter historians⁵ from proclaiming triumphantly:

Henceforth, Greek trigonometry was truly established. It was based on....tables rigorously computed. Its main object of study was always the sphere to which Menelaus' theorem applied particularly well. This theorem...paved the way for the later appearance of the sine.... The main step had been taken, and the successors—Hindus, Arabs, Europeans—had simply to follow along the trail which the Greeks had blazed for them.

According to this model, information was transmitted in bits and pieces; proofs of individual theorems were copied here and there, but Euclid's deductive method as a whole remained a monopoly of the Greeks. At the peak of racist history, colonial historians seem to have visualized information flows as follows. They imagined a 'native' who might chance upon an abandoned gun. The ordinary 'native' might worship the gun without understanding its significance. A slightly smarter 'native' might even understand how to use the gun. But the method of manufacturing guns remained a monopoly of the Europeans.

2.2. The direction of information transfer

The idea that information was transmitted, either unidirectionally, or more probably, from winners to losers in a military engagement sounds silly the moment it is explicitly articulated; it is as contrary to observation as Keynes's 'trickle-down' model of development, which enables surplus to be sucked upwards. To have systematic warfare and military conquest (as distinct from feuds between neighbours) the economic conditions must be there for it: say, a long-term period of stable surplus, leading to population growth, followed by a sudden contraction in the surplus, forcing people to change the lifestyle to which they had become accustomed.⁶ There may be individual adventurers, but people usually do not run risks collectively except when there are compelling reasons to do so.

Hence, the aggressor would, often enough, have been the one with a lower state of development of productive forces, hence of science and culture.⁷ In these cases certainly, information transmission related to military conquests very likely took place in the reverse direction; the military victor learnt from the vanquished, though not presumably in the direct way that Rama learnt from Ravana or Yudhishthira from Bhishma. In those few cases where military victory manifestly depended upon technological superiority, there is no doubt that the superior techniques would have been kept a closely guarded secret, drastically inhibiting information flows from the victor to the vanquished.

Studying the enemy would also have made sound military and diplomatic sense to those striving for conquest. Concrete examples are the people deputed by Alexander to gather knowledge for Aristotle, and Al-Bīrūnī deputed by Mahmud of Ghazni to gather knowledge about India. While the ruled could maintain a distance, as in Egypt (or, for that

matter, in any modern organization), to rule successfully the rulers had to learn about the foreign populace over whom they ruled. In short, those seeking to systematically extract surplus from foreign sources must first systematically extract information. This was true also of the European colonists, and it continues to be true today as a general proposition. Therefore, also, it is rather more probable that aggression leads to the transmission of information mainly *towards* the aggressor.

2.3. Selection effects

Whether or not it is dead, the colonial model has left behind not only bogus 'trickle-down' theories of information transfer but also a legacy of selection effects. What is a selection effect? If we focus attention on the stars in the sky at a small angular separation, they may seem related even though they are separated by vast tracts of space. If we pick stars at random from the sky then any apparent relation between the stars in the sample is likely to be purely a figment of the imagination, an artefact, a consequence of the way our attention was focused. The problem is that the case for a relationship can always be argued, for ultimately we have no means of establishing whether the stars really are related or separated.⁸ By focussing attention selectively, a selection effect can also be used to establish the *absence* of a relationship. The situation is reminiscent of the fable of the four blind men and the elephant; but a selection effect is something more than an inadvertently biased sample. It is better illustrated by the real story of the four learned men and the Indian elephant.

2.3.1. *The Indian elephant*

The facts are as follows. A piece of Mayan architecture from Central America distinctly resembles an Indian (note Indian!) elephant.⁹ Now, the American elephant became extinct some ten thousand years ago, whereas the roots of the Mayan civilization were not more than three thousand years deep. So one is naturally tempted to ask: 'What induced the Maya to sculpt Indian elephants?' The similarity of the Egyptian and Mayan pyramids is well known, and is suggestive of organized navigation between Egypt and South America. Should one combine this with the known fact¹⁰ that commerce between India and Egypt involved the Indian elephant?

But to the scholarly mind that is not the relevant question. Admitting such questions would amount to admitting the possibility that the Europeans were not the first to navigate across the Atlantic, and that would remove the last vestiges of any justification for the genocide in the Americas. Therefore, a more important issue must be settled first. What looks like an elephant to the untrained eye may or may not be an elephant—as is the case in more modern art. Here is a summary of the scholarly controversy that erupted in the well-known journal *Nature*.¹¹

Professor Tozzer bases his views on the fact that the Maya also sculpted the macaw—a long-tailed, brightly coloured parrot that is native to South and Central America. Accordingly, he holds that a comparison of the 'elephant' with the unmistakable sculpture of the macaw 'shows that the two represent the same animal'. What seems to be the elephant's trunk is no more or less than a stylized depiction of a macaw's beak.

Professor Elliot Smith suggests, 'The accurate representation of the Indian elephant's profile, its trunk, tusk, and lower lip, the form of its ear, as well as the turbaned rider and his implement, no less than the distinctively Hindu artistic feeling in the modelling are entirely fatal to the macaw hypothesis.'

Dr Eduard Seler's view is that the objects under discussion are tortoises. Disagreeing also with those who have favoured the tapir, Dr Spinden is quite definite: 'That the hands with projecting snouts, used as architectural decorations, are connected with the concept of the snake rather than the elephant is easily proven by a study of homologous parts in a series of designs.' The four learned men did not exhaust the possible interpretations of the piece of sculpture; presumably Erik von Daniken would interpret the object in question as a representation of an astronaut wearing a sulphur-dioxide mask!

2.3.2. Selection effects and transmissions

As an illustration of a selection effect, in our context, consider the point made by Srinivas.¹² The texts chosen for translation were the authoritative ones. The custom was for authoritative texts to lay out things with the utmost brevity. Methodological explanations were relegated to less authoritative commentaries, which hence remained untranslated. Consequently, so little is known about the methodology that European historians of science happily concluded that results in Indian mathematics were derived without any method!

For example, Filiozat states:

[Aryabhata 1] obtained the value of π as follows: 'The circumference is 62,832. The diameter is 20,000. By this rule the relation of circumference to diameter is given.' Its value works out at 3.1416. . . this was an example and no general rule. . .¹⁵

The quote continues by approvingly citing an example of relatively trivial 'general laws' formulated by Aryabhata: 'One should subtract the sum of the squares of two factors from the square of their sum; half the result is the product of the two factors.' Or as we would put it:

$$ab = \frac{(a+b)^2 - (a^2+b^2)}{2}$$

On the other hand, from the viewpoint of the epistemology of current formalistic mathematics, methodological issues are of the greatest importance, and there can be no mathematics without methodology. Thus, this selection effect enables what has been called the classical trajectory of the development of mathematics¹⁴ which starts in Greek times, jumps to the Renaissance, and continues down to the present time, so that any mathematics done anywhere else at any other time is reduced to a historical curiosity that cannot be linked to current mainstream mathematics, barring a few items such as Arabic numerals and the zero. This shows how a selection effect can enable one to see a direct relation between two mathematical developments widely separated in time, thus enabling one to ignore everything else in between.

It is certain that in the calculation of π the dividing line between geometry and analysis had definitely been crossed by the time of Madhava of Sangamagrama (fourteenth century) in Aryabhata's school, by continuing Aryabhata's techniques to derive the central result of the differential calculus relating to what is today called the Taylor's series expansion. But because of the selection effect, the growth of information between Aryabhata and Neelkantha Somayaji went unnoticed for long, since Neelkantha was seen as a mere commentator on Aryabhata, the true authority. Hence, also, it has gone unnoticed that the calculus was very likely transmitted from Kerala to Europe through widely disseminated pedagogical manuscripts such as Jyesthadeva's *Yuktibhāṣā* (AD 1530), probably acquired by

Jesuits. This shows how selection effects may mask transmissions, establishing the absence of a relation where a continuity is manifest.

2.4. Epistemological issues

We saw above one selection effect, viz. the selective exclusion of methodological tracts due to a clamour for authority. This selection effect operates by applying a standard epistemological filter to cloud alternative epistemologies. The situation is confounded by the historians' inability to comprehend or otherwise to come to terms with the terseness culturally required of authoritative pronouncements. Hence, it is difficult to answer questions of information exchange without reworking the entire epistemological foundations of traditional mathematics. Clearly, the way one values Brahmagupta's equation $0/0=0$ depends upon whether one regards it as valid or invalid. Current-day mathematicians have almost unanimously¹⁵ rejected this equation as erroneous because of a rather naive belief in the universality of the definitions of symbols such as 0, /, =, together with an uncritical acceptance of the universality of the Archimedean property, the current number systems used in mathematics (as distinct from those used in computation), and of 2-valued logic. That epistemological issues are the key to exchanges is also apparent in the following discussion.

2.5. Non-transmission of information

In Western histories of science, a key reason for the interest in establishing transmission has been the theological interest in justifying exploitation. The theological justification for colonial exploitation needed transmissions, real or imagined, because it appealed to the clerical principle of priority.¹⁶ The argument adopted was simplistic: if there is similarity and if there was contact, then there must have been transmission. (Thus the natural thing to do was to look for a similarity between any proof of the 'Pythagorean' theorem, and the 'proof' of it in the *Elements*.) The habitual and unreflective use of this argument has practically made cultural contact synonymous with transmission. Undoubtedly, there are some cases where this argument did apply: for instance in the case of large-scale transfer of information from Egypt to Greece.¹⁷ But the clerical apologists did not apply it in this context. On the other hand, this argument is inapplicable to the case of information flows into older and more insular cultures. This was the case where the argument was applied. Hence it is necessary to emphasize that contact only provides the potential for information exchange; contact is not synonymous with information transmission for there may be contact without information exchange.

If clerical apologetics for surplus extraction is not the goal, then it is clear that cases where information was NOT shared, despite extensive contact, are equally interesting. But this situation seems never before to have been studied in detail. There are many such cases where the information was freely available and was simply not used.

One example is that of Jai Singh. He studied all the available systems, from the European to those of Ulugh Beg, but did not incorporate the knowledge of, for example the telescope in his design of the Jantar Mantar in Jaipur. Jai Singh certainly knew about the telescope. He had bought one at a cost of Rs 100, and had used it to observe 'bright stars in broad daylight—say around the noon hour.' He had observed 'the planet Saturn..Jupiter', and knew that 'the Sun rotates...on its axis...' He had recorded these observations in his *Zij Jadid Muhammad Shahi*.¹⁸ However, he did not incorporate the telescope in his design of the Jantar Mantar, 'since the telescope is not readily available to an average person.'¹⁹

Jai Singh clearly seems to have regarded knowledge, in general, and the telescope, in particular, as only a means to an end that was partly pedagogical in this instance. Hence he rejected what would today be regarded as 'superior' knowledge. There are other cases of non-transmission that are so long lasting that they cannot conceivably be put down to any individual aberration or idiosyncrasy. I will take up two such cases. The first concerns Euclidean geometry and the second concerns the calculus.

While there is some doubt whether 'Euclidean' geometry is at all a Greek tradition, there is no doubt that Euclidean geometry is not solely a Greek tradition. It was very much in vogue in the eastern parts of the Roman Empire, and among the Arabs and the Mughuls. Abul Fazl learnt Euclidean geometry, in India, presumably from Arabic and Persian sources, and mentions it in detail in the *Ain-i-Akbari*.²⁰ India had contacts with the Greeks certainly since the time of Alexander. There were extensive trading contacts with the Roman Empire. Nevertheless, the influence of Euclidean geometry is not traceable in the writings of non-Muslims in India until Kamalakara, Jehangir's court astronomer, after the arrival of Jesuit priests in Akbar's court. Though mathematics, we are told, is one and universal, there were two streams of geometry simultaneously prevalent in India. Eventually, parts of the *Elements* were translated into Sanskrit only in AD 1718, well after the arrival of the Europeans.

A similarly negligent attitude towards Euclid prevailed among the Chinese whose geometry was tied to practical concerns, and did not pay much attention to the Platonic ideal of theoretical demonstration or 'proof' so popular with medieval European rational theologians and historians of science. The earlier rational theologians of Islam gave this ideal of demonstration a Neoplatonic twist, as we shall see later, and, as expected on the above-mentioned theory of transmissions, this Neoplatonic version travelled towards the aggressor from Mongolia, after the fall of Baghdad. One finds in Needham²¹ that it was only after the thirteenth century that:

Yang Hui...proceeded to give a proof about parallelograms which is similar to the one in Euclid. If such proofs had been extended the Chinese might have developed an independent deductive geometry, and clearly some minds like Yang Hui were prepared to appreciate the Euclidean system. This is of great interest because there may have been at this time a translation into Chinese of Euclid's *Elements*, due to Chinese-Arabic contacts.

Why did these two older cultures not share the Western historian's enthusiasm for Euclid's *Elements* and the deductive method? This is a key question because of the central role of Euclid not only in the Western scheme of the history of mathematics, but also as a model for modern mathematics. The non-transmission of information about the *Elements* between the Arabs and the Indians thus emerges as a key fact which goes against the entire scheme of transmission in the Western history of mathematics, and also the current belief in the 'universality' of mathematics used in the foundations of modern mathematics.

The final example of non-transmission concerns the calculus. The accounts that it was invented by Newton or Leibniz are certainly exaggerated, if not downright deceitful, like Brewster's biography of Newton, which has only recently been partly amended.²² It has been suspected for some time that the calculus was discovered in India, and accounts of this possibility have appeared in the popular literature.²³ This possibility is now a certainty. The calculus originated in the Kerala school of mathematicians who followed Aryabhata. The technique central to Newton's *Principia* (and to modern analysis) was the series expansion, usually attributed to Brook Taylor, a pupil of Newton, whose paper on it dates from AD 1715. Madhava used the same expansion for the sine, cosine, and arctan functions in arriving at his accurate value of π . Details of the work were already in wide

circulation in Malayalam at about the time that Vasco da Gama landed in Calicut. The Jesuits must have known about it.²⁴ Unlike the Arabs who faithfully attributed to their sources even more than what they had received, the Europeans attributed nothing at all to their sources; not a single European during the Renaissance credited any mathematical or astronomical technique or discovery to an 'oriental' source, though hundreds of thousands of manuscripts were acquired. But the question here is not how information about the calculus was transmitted to Europe without acknowledgement. The question is why this information was *not* transmitted to others. Prima facie, this information seems not to have reached even Delhi or Benares until quite recently, after a gap of some 500 years.²⁵

Perhaps there was a language barrier. But this is not an adequate explanation, since the *Yuktibhāṣā* had already been translated into Sanskrit. Moreover, Madhava's accurate table of sines must have been useful for navigation, and this perhaps did not even diffuse to the Mapilas on the western coast of India, in the same locality, who certainly spoke Malayalam, even if they used the Arabic script. In my view, this non-transmission indicates a disruption of the prevalent environment of information sharing, before it was finally dismantled.

Three stages and kinds of the non-transmission of knowledge are thus visible. At the first stage, information flowing in is critically evaluated and some of it is rejected because of epistemological differences; the example here is the *Elements*. At the second stage, the traditional information-sharing network is disrupted, and information preferentially flows out; the example here is the calculus, represented here by computations of the value of π . At the third stage, despite conscious efforts at information gathering, the information actually brought back is of such poor quality that it is rejected; the example here is Jai Singh and the information on European astronomy which he got from the Jesuits.

2.6. Cooperative versus competitive models of information sharing

Lastly, we must also examine the way in which information was shared within the society, for this also decided the sort of information that could or could not easily be transmitted to others.

The model of information sharing current in our civil society today is a 'competitive' one. The belief is that individuals (or small groups) create information.²⁶ The information so created is privately owned by the concerned individual, who as its creator acquires a right to 'royalty' or 'copyright' or 'patent', i.e. a right to extract surplus from others with whom the information is shared. In principle, the society recognizes the creativity of the individual, and encourages it by enabling creative output to be swapped for a more dominant position in society. In the case of conflicting claims, ownership is decided by priority, and in the case of conflicting claims of priority, priority (hence ownership) is decided by authority, including judicial and historical authority. In modern industrial societies, ownership of information is highly valued, and so also is technological innovation (which can lead to dramatic increases in the efficiency of production).

This was not the situation in more traditional societies where, to give an analogy in terms of land-ownership patterns, there were large common spaces. Traditionally, creative activity was seen as the manifestation of an immanent God, quite distinct from Augustine's transcendent disciplinarian. Laws and traditions restricting the sharing of information related to the sharing of religious rather than secular information; these restrictions typically applied to whole groups (say castes, foreigners, etc.). There were some conventions of apprenticeship, such as the tradition of the guru-shishya or the ustād-shāgird. These regulated information flows in the manner of the religious techniques of initiation, rather than

the commercial sale of property. Thus, while specialized information of immediate economic importance continued to be kept a secret within families and guilds, there were no laws governing its sharing and no priority disputes. Identifying oneself as the author of an innovation was not, therefore, terribly important as it was to Newton and Leibniz who quarrelled with each other so nastily²⁷ over priority for the calculus, which neither of them had. Value was attached to 'authority' and the age of a tradition; tradition could be rejected in favour of a better system (as for example Varahamihira did, while updating the *Vedanga Jyotisha*), but innovativeness was not valued for its own sake.

Thus, in this 'cooperative' model of information sharing, information might be held in secret for its economic value, and information might not be given out if the recipient of the information was not regarded as worthy enough to receive it. But information was not held in secret merely for the sake of establishing one's innovativeness to posterity—it was unimaginable that someone would threaten, as Newton threatened Hooke, to withhold publication to demonstrate priority. Indeed, if someone did make a small innovation which he regarded as valuable, it could well go in as an anonymous contribution to a book being copied out or commented upon. This was particularly true of Arabic traditions, where the numerous translations were never mechanical. Sailors' manuals attributed to ancient sources could thus contain up-to-date information.²⁸ Similar processes must have been active in the case of Euclid's *Elements*, which show epistemological fissures, as we shall see later.

A concrete model of this sort of information sharing can still be seen in remoter places such as the Lakshadweep islands. The result is quite striking. Though there is undoubtedly a common pool of information, there is also differentiation. Without any active attempt to keep anything secret, I found that islands which are only 30 km apart can have discernibly different traditions, and may be unfamiliar with some of each others' navigational instruments.²⁹

To summarize, in this model of information sharing, it is neither possible nor important to try and trace each key development to an imagined unique source from which it diffused. To use an analogy, in locating the origin of agriculture in the Fertile Crescent or somewhere else, we are modelling information flows by a river which has a source. This may be true of some sorts of information flows; there are rivers, but there is also the sea—of shared information—for which it is futile to seek a source. In this case, it is more interesting to look at currents and waves—individual peaks of localized information that only emphasize that it is the peaks that need an explanation rather than the flat background of a very large shared base of common knowledge due to extensive contacts.

This cooperative model of information sharing was disrupted with the arrival of the Europeans, who systematically attempted to localize information by establishing asymmetric information flows towards themselves, in the manner of dams across rivers.

3. THE CHANNELS OF INFORMATION TRANSMISSION

In contrast to models of information sharing, which have been neglected by scholars, *channels* of information transmission have been fairly well studied. Military or commercial exchanges created channels along which information could easily flow. India was connected to China and West Asia through both land and sea routes.

The land routes have been extensively documented.³⁰ Trade routes have existed from before recorded history, among the most famous being, of course, the Silk Route. Aggressors

with large empires who sought to extract larger volumes of surplus from far-off lands were forced to maintain the land routes. Examples are Alexander who had to link Greece to Afghanistan; Kanishka who linked Central Asia, West Asia, and north India; and the Mughul Empire in Baghdad which linked Central Asia and West Asia. I do not need to go into these details, since they will no doubt be covered in other articles in this volume.

There are, however, three points that I would like to emphasize. The first concerns the bandwidth or the potential volume of information transmission. One would expect this to be proportional to the volume of trade, or the forcible surplus extraction, and this is usually underestimated. I would, therefore, like to draw attention to Pliny's complaint about the reverse extraction of the surplus that in no year did 'India absorb less than five hundred and fifty million sesterces of our empire's wealth, sending back merchandise to be sold with us at a hundred times its prime cost.'³¹

The second point that I would like to emphasize is that, analogous with the Internet (or a general packet-switched network), the route of information transmission need be neither unique nor the most direct one. As a concrete example, consider the *meruprastāra* in Pingala's *Chandahsutra*. It could have first travelled to China, and from thence to Europe through Jesuit intermediaries, where it eventually came to be known as Pascal's triangle, giving the coefficients of the binomial expansion, attributed to Newton.

Thirdly, for the purposes of this article, the sea routes are relatively more interesting, for navigation involved the practical application of both astronomy and mathematics; it also provided a context in which information had to be shared, and tradition certainly would not have stood in the way of any technique which manifestly fetched results. The sea routes were presumably used to carry heavy cargo, like the Indian ebony that was exported to Rome.

Navigation did exist; the idea that early navigators simply crept along the coast is certainly false. This was the method adopted by Vasco da Gama, who was ultimately compelled to accept the advice that this technique could only lead him to the Red Sea and that he had to strike out across the ocean to get anywhere near the source of the spices. The 'Guzerati Moor', Malemo Cana³², who brought Vasco da Gama to Calicut, certainly understood how to navigate across the sea from Africa to India. A variety of islands were known: Mahal Dvipa (Maldives) is found in every Arabic mariner's manual. Sailing out to these islands certainly involved sailing out of sight of land; hence it necessarily involved sharp problems of navigation, for small islands can easily be missed, as is recorded in European navigation manuals of even the nineteenth century. From the earliest times, islands such as Lanka and Java were known.

It has been alleged that though Indo-Arabic contacts stretch back to antiquity, the volume of sea traffic was small until Hippalus. The 'discovery' of the monsoon winds by Hippalus seems inauthentic, and is perhaps a product of the historians' imagination like the 'discovery' of India by Vasco da Gama. Perhaps this really was a discovery for the Romans, who learnt of navigation rather late. In fact, Alexander sent one of his generals on a sea voyage to learn about the sea route to India.³³ Kautilya mentions the appropriate times for crossing the sea, suggesting that this was an established routine by his time.³⁴

The sea route also extended to China. As described by Fa Hian,³⁵ on his way back he stayed out at sea for a rather long time of ninety days. From Fa Hian's account it would appear that people could not navigate when the sky was overcast. This suggests the inference that the magnetic compass was not then in wide use, and that the navigational techniques in general use then were purely celestial. It should be pointed out here that navigational problems were particularly acute on the eastern coast of India due to the erratic monsoons, sudden shifts in wind direction, and a practically east-west course. There

are also problems associated with uneven sea depth, sunken reefs, and magnetic anomalies. These acute navigational problems faced by the Cholas may have been part of the reason for wanting better sine values in astronomical techniques of navigation.

Most of the early navigators seem to have been perfectly aware that the earth is round. Certainly, scholars from Aryabhata³⁶ and Varahamihira³⁷ to Al-Bīrūnī³⁸ not only stated this but they also arrived at fairly accurate estimates of the radius of the earth. (They erred in taking the earth to be a perfect sphere.) These estimates were far superior to the European estimates from Columbus to Newton. European sources must be considerably biased, for they attribute to Ibn Majid, five centuries after Al-Bīrūnī, a knowledge of trigonometry far inferior to that possessed by the former; Tibbets³⁹ contends that Ibn Majid thought that two sides of a triangle add up to the third! Perhaps the sixteenth century witnessed a general decline of the Arabic civilization; nevertheless, it is mysterious that this alleged ignorance was maintained despite extensive contacts with India. Being relatively isolated, the traditional navigational techniques of the Lakshadweep islands give us a fairly clear picture of the navigational techniques then used in the Arabian Sea. This knowledge of the round earth and its size was embodied in the traditional navigational practices: in the definition of the *zam* as the 'distance from here to the horizon.'⁴⁰

4. MATHEMATICS

The contours of the basic story of cultural interactions and exchanges are well known. Information sharing between India and China became prominent shortly after the arrival of Buddhism in China, in the first century CE. Some important subsequent exchanges concerned logic, numerals, especially the symbol for zero, the rule of three, and calendar making and all the mathematics associated with it. Information sharing between India and Central and West Asia first became prominent at the time of Kanishka. I am not aware of any detailed documentation of exchanges in mathematics during this period. Interactions then became prominent with the rise of the Arabs in Baghdad. At this time it is known that a number of Indian manuscripts in mathematics and astronomy were translated into Arabic. Al-Bīrūnī was among those who continued this process of translations. It was during this period that *sunya* was transmitted as *as-sifr*, then cipher, zephyr, zero, and *jivā* was transmitted as *jibā*, (opening), *sinus* (fold). About a century after the fall of Baghdad to the invading Mughuls, the information sharing between West Asia, Central Asia, and China peaked.

As already explained above, I will be concerned not with a more potent rehash of this mainline story, but with examining certain key silences in it. One sort of silence is that due to oversight. Dinnaga's treatises on logic are available to us only in Tibetan and Chinese translations, and logic is at the foundation of mathematics. But no history of mathematics has, to my knowledge, examined the exchange of principles of logic between India and China, via the Buddhists. Again there is the exchange of modular arithmetic—the computation of what is called by the historians of mathematics as the residual pulverizer, and by mathematicians as the Chinese remainder theorem. Altogether another sort of silence is that due to implicit assumptions about information exchange. After examining these assumptions above we settled on three cases of non-transmission as important counter-examples that invalidate the usual assumptions. Two of these cases of non-transmission concern mathematics: Euclidean geometry and the calculus.

4.1. Non-transmission of Euclidean geometry

4.1.1. *The key to geometry*

As remarked earlier, Euclidean geometry is also a strong Arabic tradition. There are at least three dozen known commentaries and translations of the *Elements* in Arabic, including those of al Kindi, Thabit Ibn Qurra, al-Farabi, al-Haitham, Ibn Sina, and Nasiraddin at Tusi. There is little doubt that the Arabs got their knowledge of geometry from outside; the Haji Khalfa records that Caliph al-Mansur (754–75 CE) sent a mission to the Byzantine emperor, from whom he obtained a copy of the *Elements* among other Greek books. Caliph al-Mamun (813–33 CE) similarly obtained a copy of the *Elements* from Byzantium. According to the *Fihrist*, an Arabic index of books, al-Hajjaj translated it twice, in a 'Haruni' (for Harun ar-Rashid, 786–809 CE) and a later 'Mamuni' version. The Arabs referred to Euclid as Uclides, which they derived from *Ucli*, a key, and *des*, measure or particularly the measure of the earth (= geometry), so that *Uclides* meant *key to geometry*.⁴¹ The Arabic translations are not mechanical; they did not treat the text as sacrosanct, and felt free to make improvements. The available Greek texts of the *Elements*, which have been dated to as far back as 888 CE, may be younger than some of the Arabic ones, and it is clear that they too did not treat the text as sacrosanct. Hence it is not particularly clear that the Greek texts are more authentic. (Nor even is it clear what exactly this authenticity means.)

4.1.2 *The epistemological barrier*

In India, this Arabic tradition did not mix with the indigenous tradition of geometry. There remained two streams of geometry in India, an Arabic one used by Muslim scholars such as Abul Fazl, and an indigenous tradition dating back to the *sulba sutra*. This non-mixing cannot be put down solely to insularity; after all there was a mixing even in religious matters, visible in the Sufi and Bhakti traditions at the ground level. I will argue that the non-mixing was the result of important epistemological differences, which created an epistemological barrier to transmission.

Details of the differences between the Indian tradition of geometry and the *Elements*, though quite basic, are a little confusing since they are typically seen from a third epistemological perspective, that of the formalistic modern mathematics of Hilbert and Bourbaki, which itself traces its inspiration to the 'Euclidean' model of geometry. The confusion sown by the historians of mathematics has clearly influenced most mathematicians in the country, since the confusion is still visible in our current school syllabus for geometry. Hence this point needs elaboration: this variety of epistemological perspectives is a barrier to information transmission today!

4.1.3. *The synthetic versus the metric approach to Euclid*

Western historians of science have clearly invested centuries of effort into studying Euclid. This effort ultimately paid off when, at the end of the nineteenth century, mathematicians such as Hilbert and Russell got involved in restoring the foundations of deductive geometry. In fact, it is well known that the present formalistic epistemology of mathematics (axiom-definition-theorem-proof) can be traced back to the efforts, notably Hilbert's, on the foundations of geometry at the beginning of the twentieth century.

In trying to clarify the notion of 'proof' in Euclid, Hilbert was faced with the following difficulty, which must have been noticed by every schoolchild who studied geometry using the older texts, current in India until the 1960s. Proposition 4 of the *Elements* states that if two sides and the included angle of one triangle are equal to those of another triangle,

then the two triangles are equal. In the *Elements*, the proof of this proposition involves picking up one triangle, moving it through space, rotating it as necessary, and applying it to the other triangle. The doubt that must have entered the mind of every schoolchild is this: this method greatly simplifies the proofs of all other theorems; if it can be used in one place, why cannot it be systematically used in other places as well? My teacher had no satisfactory answer to the query why it was all right to do this in one place, but wrong to do it elsewhere. But one may attempt an answer as follows.

Picking and carrying line segments is a common enough thing; one necessarily does it every time one makes a measurement. But mathematicians were sceptical about the very possibility of making a measurement: what sense did it make to say that a figure remained identical to itself as it was moved about in space? Consider, for example, one's shadow relative to a far-off light source like the sun. As one moves about, one's shadow moves on the ground. Movement over an uneven surface clearly distorts the shadow. The same thing may happen to a line segment moved through a space which is uneven. So, how does one know that figures will remain invariable when moved? How does one know that space is homogenous? One can, of course, *define* the meaning of rigid motion—a thing remaining congruent to itself as it moves in space—in such a way that it agrees with some intuitive ideas. And this was what Birkhoff did in his metric approach to Euclidean geometry. This may be called the B-revision of the *Elements*.

But this meant the loss of valuable ground to the Western historian of science. If the superposition of figures was allowed as a method of proof, the proofs of most propositions (for instance Propositions 1, 2, 3) in the *Elements* would be trivialized. Bertrand Russell remarked that Euclid would have been better off accepting Proposition 4 as a postulate, as Hilbert had practically done. The US School Mathematics Study Group (MSG) arrived at the same recommendation in 1957, and our current school syllabus follows the strategy they laid down, so that Proposition 4 of the *Elements* is now known to every schoolchild as the SAS (Side-Angle-Side) postulate. This may be called the H-revision of the *Elements*.

The H-revision hopes to salvage the deductive method by retaining some need to prove the remaining propositions in the *Elements*, while allowing that Euclid had made an error regarding Proposition 4. This approach to geometry, which does *not* involve measurement, has hence been called the synthetic approach. The most graphic description of this approach is through de Morgan's 'collapsible compasses': distances cannot be picked and carried, because the compasses are loose and 'collapse' as soon as they are lifted from the paper. In this synthetic approach, the use of the term 'equal' in the *Elements* is re-interpreted as *congruence*, for both angles and segments. All this is, of course, quite elementary and well known.

4.1.4. The end of the synthetic approach

The point which has not been sufficiently highlighted is this. *The synthetic approach simply cannot proceed beyond the first 34 propositions in the Elements.* Proposition 35 states:⁴² 'Parallelograms on the same base and in the same parallels are equal to one another.' Here, 'equal' cannot possibly be interpreted as congruent, because the two parallelograms in question may clearly be taken to be incongruent. (Fig. 1) The *Elements* does not state what 'equal' means. 'Equal' here presumably means 'equal in area'. *But this semantic metamorphosis demolishes the*



FIG 1. Proposition 35 of the *Elements*

This proposition declares two incongruent parallelograms 'equal' without defining area

synthetic approach, for the notion of incongruent figures with equal area requires measurement. This particular proposition is no interpolation; Proclus specifically comments on this proposition as being surprising and paradoxical. In the H-revision of the *Elements*, i.e. in the synthetic approach, it does not help to try to define a general area through triangulation, since the usual formula for the area of a triangle = $\frac{1}{2}$ base \times height derives from this Proposition 35, by regarding the triangle as half of a parallelogram which is then equated to a rectangle, whose area is presumed to be known. Nor does it make much sense to define area as a measure, without first defining a metric: the integral connection of the Lebesgue measure with the metric topology is too well known to be reiterated here. In ordinary language it is utterly senseless to define *area*, without first defining *length*.

4.1.5. *The non-definition of area in the Elements and its consequences*

Since measurement is not defined anywhere in the *Elements*, neither is area. Nevertheless, there are other key propositions which make use of a notion of area. (Geometry, after all, literally means the measurement of the earth.) The most famous of these propositions is the so-called 'Pythagorean theorem', which appears as Proposition 47 in Book I of the *Elements*.⁴³ 'In right-angled triangles the square on the side subtending the right angle is equal to the squares on the side containing the right angle.' Without defining area, it is entirely meaningless to talk about [the area of] one square being 'equal' to the 'sum' of [the areas of] two squares, for the 'sum of two squares' need not even have the shape of a square. The pretence of a strictly deductive approach is maintained only through the trick of changing the meaning of the term 'equal' halfway through the book.

4.1.6. *The central paradox of the Elements*

We now see the paradox in its fullness. If the synthetic approach is used to reconstruct the *Elements*, as in the H-revision, all propositions from Proposition 35 onwards must be abandoned as not only unproved but also meaningless, and that includes the 'Pythagorean theorem'.⁴⁴ If the metric approach is used, the proofs of most of the propositions in the *Elements* are trivialized, so that what we have here is not that monument to the deductive method, so glorified by the Western historians of science, but a book which seems ritualistic⁴⁵ and asinine⁴⁶, as it did to the Epicureans. (This paradox still remains true of the current National Council for Education Research and Training syllabus for school geometry, with its SAS postulate, and its mindless adoption of the US SMSG's recommendation to use a synthetic approach.)

For example, in the metric^{46a} approach, without the aid of the preceding 46 propositions, one can straightaway prove the 'Pythagorean theorem' in two simple steps, (Fig. 2) as in the *Yuktibhāṣā*. Though the core idea of this proof is the same as the one attributed to Thabit Ibn Qurra by an-Nairizi,⁴⁷ we see that the epistemological bases of the two proofs are quite different, so there is no reason to believe that the similarity is due to any transmission.

The shift in the epistemological base, from synthetic to metric, halfway through the *Elements* shows that, like Humpty Dumpty fallen off the wall, the H-revision of the *Elements* is cracked down the middle. Will the efforts of historians and mathematicians ever put it together again? Even if they do discover a way to do this in the future, it is clear that Western historians and mathematicians have, for centuries, been praising something sky-high without understanding it!

4.1.7. *The epistemological basis of the sulba sutra*

As the name suggests, the epistemological basis of the *sulba sutras* is metric: the *sulba* or rope was used for measurement. The geometry of the *sulba sutra* has been described as

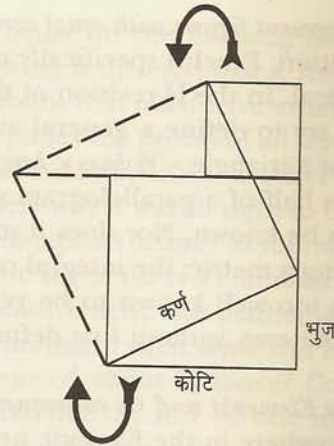
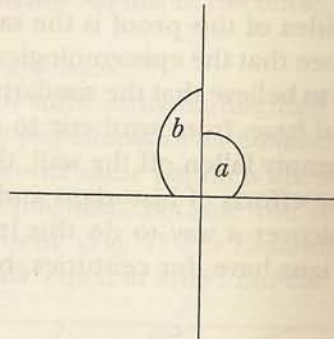


FIG 2. The Yuktibhâsâ proof

Place the smaller square as shown. Measure the Koti from the NE corner (or the bhuj from the SE corner), cut along the karna-s, and rotate to form the square on the karna.

'ritual geometry'. It would be more correct to call it 'practical geometry', for while the altars may have been used for ritual purposes, the actual construction of the altars was a practical matter. The practical approach is also evident in the definition of a right angle, using a plumb line. To those educated in the modern mathematical tradition, which glorifies abstraction and the deductive method, this may sound a bit gross, but the plumb-line definition is certainly easier to understand than the definition of 'right angle' in the *Elements*, which involves yet another play on the word 'equal', in the sense of 'equal angles' defined (without defining how to measure angles) in Definition 10:⁴⁸ 'When a straight line set up on a straight line makes the adjacent angles *equal* to one another, the adjacent angles are called **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.' Without an a priori notion of what a right angle is, this situation is not greatly clarified by Postulate 4 which states that 'all right angles are *equal* to one another'! It may be possible, following Hilbert, to interpret this particular use of the term 'equal angles' as 'congruent angles' so defined that this makes sense, but it is equally possible to interpret it so that it does not make sense (Fig. 3). No sanctity need be attached to Hilbert's interpretation for, as we have already seen, it cannot be the correct one, for it does not give any meaning to the later parts of the *Elements* from Proposition 35 onwards.

FIG 3. The right angle in the *Elements*

By Definition 10, if the two angles are equal then they are right; but by Postulate 4, if the two angles are right then they are equal! So does $a=b$?

4.1.8. *The Elements and rational theology*

It is well known that a key reason for glorifying the *Elements* was the medieval Thomist scheme of rational theology, which sought to prove to the non-believer the existence and uniqueness of God through rational arguments. (It is also well known that in modern formalistic mathematics the main concern of a large body of theorems is to prove existence and uniqueness; where such proofs of existence and uniqueness do not exist, as in the axiomatic quantum field theory, the theory itself is regarded as suspect and non-rigorous.) However, it is not so well known that the medieval rational theologians derived much knowledge and inspiration from the Arab rationalists (the Mutazilah).

We now see that the *Elements* exactly fitted in with their aim of rationally deducing everything from the two premises of equity and divine justice, for clearly 'equality' is the key notion that is taken as logically primitive in the *Elements*: equality of angles, equality of segments, and equality of areas. In fact, the *Elements* fits in so well with the scheme of the Mutazilah that this casts serious doubts on the early Greek origin of (the current form of) the *Elements*.⁴⁹ At any rate, the *Elements* was available to India and China mainly in the Arab form, and it is clear that this form that the *Elements* must have been influenced by the Mutazilah notion of equality. This may have decisively influenced their non-transmission as follows.

In today's world, we find that physicists and engineers are concerned with calculations and numerical computations, but are unconcerned with 'proofs' which they regard as rituals peculiar to mathematicians. Undoubtedly, similar considerations must have acted against the acceptance of the *Elements*; for the theorems useful for calculations were already long known, and the special feature of the *Elements*—the scheme of rational deduction from a logically primitive notion of 'equality'—may well have been seen as theologically and ritualistically motivated. (Proclus, the earliest actual source of the *Elements*, takes great pains to emphasize the linkages of mathematics and the *Elements* to Neoplatonic theory and practice.) There may well have been the added sociological dimension that a logically primitive notion of equality may have been socially unacceptable, but I will not go into this dimension here.

4.1.9. *The logic underlying deduction*

There is the further feature that the deductive scheme in the *Elements* appeals to Aristotelian logic. Intercultural variations in logic are particularly interesting. For the Arab rationalists (like the Mutazilah), who mistakenly identified Aristotle as the author of the *Enneads*, rationality meant Aristotelian logic. This point was firmly reinforced by Thomists and later-day rational theologians, and goes without saying in modern formalistic mathematics. But the use of Aristotelian logic probably created an additional epistemological barrier because it did not fit in with the Indian schemes of valid inference. I have commented on these differences of logic in detail elsewhere.⁵⁰

4.1.10. *Summary*

To summarize, there is no existing way in which the geometry of the *Elements* can be reinterpreted within the deductive scheme of modern mathematics as a meaningful exercise. There are two possibilities. (1) The technique used to prove Proposition 4 is a metric technique; if this metric technique is accepted, then most propositions in the *Elements* are trivial, the proofs given in the *Elements* are needlessly complex, and no particular significance can be attached to the arrangement of the propositions in the *Elements*. In this case, the whole book must be abandoned; it remains, at best, a historical record of a complex way of arriving at simple conclusions. (2) If metric techniques are rejected, as in Hilbert's

synthetic approach, which reinterprets 'equal' segments and angles as congruent, and Proposition 4 is taken as the SAS postulate, as in current school geometry, then all propositions beyond Proposition 35 (including the Pythagorean theorem) must be abandoned. The *Elements*, then, provides no real proof of the Pythagorean theorem because it does not define area anywhere. In this case, the later—and more useful—half of the book must be abandoned. In either case, though its theological or political motivation of equity is no doubt a very laudable one (and one with which I fully agree),⁵¹ the *Elements* is of little value for simple practical tasks such as building altars.

In contrast, the indigenous geometry had a straightforward empirical and metric approach which was both simple and practical, and free from these unnecessary complexities which entailed no new practical consequences. This explains the non-acceptance of the *Elements* until it acquired substantial weight of authority in the Mughul court after the arrival of the Jesuits; this involved practical considerations of a different sort!

4.2. The calculation of approximate values of π in India, China, and Central Asia

Not only was mathematics practically oriented rather than theologically oriented, at least in India and China, it was also numerically oriented. Current mathematics is still proof oriented, though the ubiquitous computer may soon change this situation. (As a concrete example, recall the instance of the current situation of quantum field theory; theologically oriented mathematicians would reject it as non-rigorous even though it gives very precise numerical results that agree with the experiment to the seventh decimal place.) Hence, we examine next the origin of the infinitesimal methods of the calculus as manifested through the calculation of approximations to π that were common to India, China, Central Asia, and West Asia. This also illustrates a key epistemological difference: it brings out the role of geometry in building *practical* and *numerical* approximations. It bears restating that these approximations (rather more than 'ideal exactitude', 'the deductive method', and theorem proving) still are at the heart of *all* applications of mathematics to the physical sciences (for example, in quantum field theory, which has so far resisted Wightman's axiomatization).

The integral calculus concerns the integrals of functions, and computing (definite) integrals is equivalent to calculating the area enclosed by a plane curve. Unlike the case of Hilbert's interpretation of Euclidean geometry, which stumbles on the question of defining area, Aryabhata, in his *Aryabhatiya*⁵², defined the area of a general plane figure using triangulation. In the *Ganita* section, he first states (6) 'The product of the perpendicular and half the base gives the area of a triangle.' He then states (7) that 'half the circumference multiplied by half the diameter gives the area of a circle.' He next states (8) that the area of a trapezium is obtained by 'multiplying half the sum of the base and the face by the height.' He goes on to state that (9) 'for any plane figure find a way to fill it [using rectangles or right triangles, and sum (half)] the product of the adjacent sides to surpass the area.' He then gives (10) the value of π : '100 plus 4 multiplied by 8, and added to 62,000: this is the *approximate [asanna]* measure of the circumference of a circle whose diameter is 20,000.' This works out to:

$$\pi = \frac{62832}{20000} = 3.1416$$

The only contemporary estimate of π which is as accurate is the value of Liu Hui, using a polygon of 3072 sides. Al-Khwarizmi, in his *Algebra*, reproduces Aryabhata's values in practically the same terminology:

The other method is used by the astronomers among them; it is this, that you multiply the diameter by sixty-two thousand eight hundred and thirty two and then divide the product by twenty thousand; the quotient is the periphery.

We recall that Filliozat nevertheless described as an accident the precise value of π derived by Aryabhata I (§ 2.3.2.).

Now it so happens that about a century earlier, in China, according to Needham:⁵³

Liu Hui...by inscribing a polygon with 192 sides within a circle and calculating the polygon's perimeter, ...obtained [π =] 157/50 or 3.14. Liu Hui also gave two other extreme values, and used a polygon of 3,072 sides for his best one, 3.14159....the Greeks...had never achieved a value as accurate as this.

Around the time of Aryabhata we find attributed to Tsu Chhung-Chih a value between 3.1415927 and 3.1415926, corresponding to the approximation 355/113, as stated and verified by about 1300 CE by Chao Yu-Chhin, using a polygon of up to 16,384 sides.⁵⁴ About a century later, in India, Madhava of Sangamagrama had calculated an even more accurate value of π , described by Neelkantha, in his *Aryabhatiya Bhashya*, as follows: 'The measure of a circle of diameter 900,000,000 is 2,827,433,388,233,'⁵⁵ corresponding to $\pi = 3.141,592,653,59$ accurate to ten decimal places'.⁵⁶ The *Kriyakarmakari* gives the value $\frac{104348}{33215} = 3.141,592,653,9211\dots$, accurate to nine decimal places, while the *Karanapaddhati* gives 31,415,926,536 as the circumference for a diameter of 10,000,000,000 and explains how the following approximations may be derived from this: $\frac{3}{1}$, $\frac{22}{7}$, $\frac{355}{113}$, $\frac{67783}{21576}$, $\frac{68138}{21689}$, $\frac{408473}{130021}$, etc. A little before Madhava, we find that al Kashi (d. 1429), the director of Ulugh Beg's observatory, had calculated the value of $\pi = 3.141, 592, 653, 589, 793, 25$ accurate to 16 decimal places, in his *Risala al Muhitiyya* ('Treatise on the Circumference').

The question now is this: was there any transmission? The question is important because of the following. While increasing precision in the values of π is only a rough indicator of overall mathematical sophistication, such precise values of π ultimately concern the integral and differential calculus, and one would like to understand how the calculus developed. But one cannot readily answer this question for, on account of the selection effect, no account has been published (to my knowledge) of Aryabhata's technique (though some unsatisfactory speculations do exist). Needham provides only a diagram from which one must guess the techniques used by Liu Hui and by Chao Yu-Chhin. Al Kashi presumably pushed the earlier techniques to new limits, but detailed information about the techniques he used is inaccessible to me. It is clear that the questions being asked by contemporaries (give or take a century) in India, China, and Central Asia are roughly the same, that there is a widely-felt need for greater precision in numerical values, and that the numerical values being provided are also comparable. The differences therefore are only in the techniques.

4.2.1. Aryabhata's method of calculating π

Aryabhata's comments are brief to the point of being obscure, and this has caused some confusion: was this a method or an example? Clearly, Western historians of science culturally prefer prolix prose to the terse verse of the *sutra* tradition. Clearly, also, Aryabhata had a general technique in mind. But no account has yet been given of this general technique, at least not to my knowledge. Hence I describe below a technique, as reconstructed from an unpublished draft translation of the *Yuktibhasa*.⁵⁷ Unlike, infinitesimal techniques, there is not even a shadow of a doubt that Aryabhata *could* have used this technique for it requires only the extraction of square roots and the definition of area, both of which have

been explicitly described earlier in the *Aryabhatiya*. This also is the technique described by Neelkantha in his commentary on the *Aryabhatiya*. In fact, Neelkantha comments on the use of the term *asanna* (approximate) by Aryabhata (Verse 10 above) as follows:

Why is this near value given and the real value left out? I will explain. Because the real value cannot be given. By any measure if the diameter is measured without a remainder, the circumference will leave a remainder most certainly. Similarly, the measurement of the circumference by any unit without remainder will not measure the diameter exactly. (Whatever the unit) there will always be a remainder. Though we may carry it on indefinitely, we can only achieve smallness of the remainder, but never remainderlessness. This is the idea.⁵⁸

Moreover, the *Yuktibhasa* still is the earliest fully translated text from Aryabhata's school which concerns an explanation of the rationale, and clearly the technique described here is one which had definitely been discarded by the school by the time of Madhava. (He had the infinite series expansion of arctan). Madhava's infinite series expansion clearly takes off from the method of computing sines using finite differences which is described next by Aryabhata. Between Aryabhata and Al-Khwarizmi (d. c. AD 850) we know of no particular improvements to the value of π obtained by Aryabhata's school. Also, the technique is of independent interest; though the technique itself is geometric, it ultimately leads to a simple numerical algorithm.

By way of background, we recall that the Greeks knew of a way of approximating the circle by a polygon: inscribe a square in a circle, fill up the gaps by erecting an isosceles triangle on each side, and continue the process. In an important variation of this technique, Archimedes started with two hexagons, one inscribed and one circumscribed, and continued this procedure by doubling the number of sides at each stage. This led to a recursion formula to compute the perimeter, by alternately computing the harmonic and geometric means. If P_n and p_n , respectively denote the perimeters of the circumscribed and inscribed n -gons, then the formulae are:

$$P_{2n} = \frac{2p_n P_n}{p_n + P_n}$$

$$p_{2n} = \sqrt{p_n P_{2n}}$$

This involved the computations of square roots, and the difficulties that the West has had with irrational (i.e. non-ratio) numbers, also called 'surds' (= stupid) are too well known to be reiterated here. Archimedes used 96-gons to estimate π as lying between $3\frac{10}{71}$ ($= 22\frac{3}{71}$) $= 3.1408$ and $3\frac{1}{70}$ ($= 22\frac{2}{7} = 3.1428$).

Liu Hui's method of computing π was rather similar.⁵⁹ He used only inscribed polygons and the recursion formula:

$$P_{2n} = \sqrt{2R \left(R - \sqrt{R^2 - \left(\frac{p_n}{2}\right)^2} \right)},$$

where R , the radius of the circumscribed circle, he took equal to 1. Liu, too, started with the hexagon, which is the natural thing to do, since in this case $p_n = 1$ (since $\sin 30 = \frac{1}{2}$). Doubling to 12, 24, 48, and 96 sides he obtained his value of $\pi = 3.141024$. Apparently Liu continued this process up to a polygon of 3072 sides. I have been unable to determine the exact method used by Liu Hui in computing the square roots, which is the critical ingredient. As far as I know, no one prior to Aryabhata states a general technique for extracting square roots. The clumsy Roman numerals made ordinary addition, multiplication, and division so difficult that square-root extraction was surely a forbidding matter.

Aryabhata, however, had an elegant method (essentially the current method) of extracting square roots using the decimal place value. This method was applied to determine the value of π as follows. Aryabhata's idea was to cut out a circle from a square (Fig. 4). We reproduce this method in full from the *Yuktibhasa* commentary to bring out the flavour of the techniques used, which have not before been explained. This process relied on octagons rather than the hexagons used by Archimedes and Liu. All calculations make repeated use of the Pythagorean theorem, better renamed the sine rule, for the Indian tradition introduced and worked with sines rather than Ptolemy's chords, and the proposition in question is equivalent to the sine-formula $R^2 \sin^2\theta + R^2 \cos^2\theta = R^2$. (Alternatively, for the sake of simplicity, it could be renamed the 'diagonal rule' for in the *sulba sutra* the rule is described by linking the square of the diagonals of a rectangle to the square of the sides.)

Step 1. Construct a square with sides equal to the diameter of the required circle.

Step 2. Draw the north-south and east-west lines to form four small squares. The required circle meets the square at the four cardinal points. Draw a line from the centre to the south-east corner.

Step 3. (Fig. 5) The idea is to cut the south-east corner C along the line AB, and to repeat this process at the remaining 3 corners of the square. The requirement is that the resulting octagon should be equilateral. Alternatively, the requirement is that the line AB should be tangential to the required circle at the point where the circle intersects the line OC from the centre to the south-east corner.

Step 4. Let x be the side of the required octagon, and r be the radius of the required circle. Applying the sine rule to the right-angled isosceles triangle ABC with hypotenuse AB, we obtain the quadratic equation $x^2 = 2(r - \frac{x}{2})^2$, with positive root, $x = 2r(\sqrt{2}-1) = 2(h-r)$, $h = \sqrt{2}r$ being the diagonal of the smaller square.

Step 5. Since triangle ESC is similar to triangle ABC, $\frac{h}{r} = \frac{x}{r - \frac{x}{2}}$, so by the rule of three, $r - \frac{x}{2} = \frac{rx}{h}$. Measure out this last quantity (= CA, CB), and cut the corner. (Observe that

this is an irrational quantity that is being calculated and measured out, a process inconceivable in the synthetic reinterpretation of 'Euclidean' geometry.)

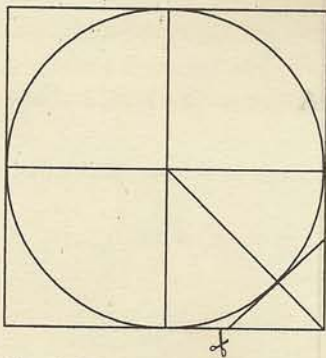


FIG 4. The desired circle
The desired circle is the one inscribed in the polygon. At each stage one cuts off from the corner of the polygon an isosceles triangle with base tangential to the circle.

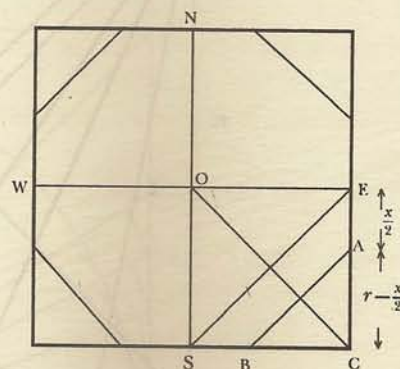


FIG 5. The octagon method

This method of calculating π starts with a square of diameter equal to that of the desired circle, and proceeds by cutting off the corners of the polygons obtained at each stage, to obtain the next equilateral polygon. This leads to a simple recursion rule which differs from the hexagon-doubling method of Archimedes and Liu Hui.

Step 6. The first approximation to the circumference ($= 2\pi r$) is $8x$, and this gives $\pi = 3.313708$.

Step 7. (Fig. 6) The idea is to cut the corner B of the octagon, along the line B_1B_2 , and to repeat this at the other seven corners to get a 16-sided figure. Observe that the required circle meets each polygon tangentially at the mid-point of its sides. Thus the line joining the centre to the mid-point of the side of the octagon has length r . Solving the right-angled triangle OBO_1 , gives $OB^2 = r^2 + x^2/4$, hence $BO_2 = OB - r$. But $BB_2 = x/2 - y/2$, and $O_2B_2 = y/2$, so we can calculate y by applying the sine rule to the triangle BO_2B_2 . In fact, this gives the formulae, $y = r \frac{a^2 - k^2}{a}$, where $a = \sqrt{2} - 1$, $k = \sqrt{1 + a^2} - 1$, and $\pi = 16k/a$.

Step 8. The method and calculations in the above step can be repeated indefinitely. Hence we are led to the following numerical algorithm. Let,

$$g(x) = \sqrt{1+x^2} - 1$$

$$f(x) = \frac{g(x)}{x}$$

The algorithm computes to level n ,

$$z_0 = a = (\sqrt{2} - 1),$$

$$z_i = f(z_{i-1}),$$

$$\pi = 2^{2^{i+1}} z_i$$

It is clear that this algorithm involves the computation of only squares and square roots, and Aryabhata had already stated efficient algorithms for these, which use the decimal place-value notation. Since our primary concern here is with interactions, rather than the algorithms per se, instead of coding Aryabhata's algorithms for squares and square roots, we took a short cut and wrote a computer programme, using the intrinsic sqrt function in Turbo C. The results show that Aryabhata used either the value $n = 5$, or the value $n = 6$, corresponding to a polygon with 512 sides, or 1024 sides.

Thus, Aryabhata's method could *not* have been the method used by Liu Hui, who clearly used a technique similar to that of Archimedes, since $3072 = 3 \times 1024 = 3 \times 2^{10}$ is not

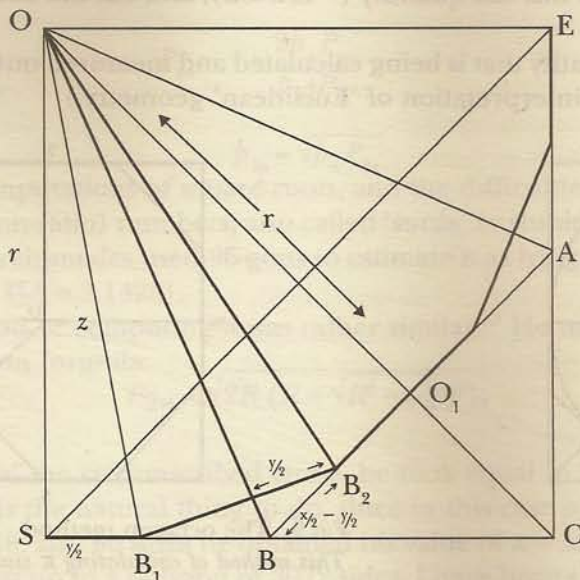


FIG 6. Part of the 16-gon in a small square

The figure shows the situation in the south-east square where two corners of the octagon at B and A are cut, along a tangent to the circle. The key to the recursion of formula is that the required circle meets each polygon tangentially at the mid point of its sides.

a power of 2 but is a number that would be obtained on the hexagon-doubling method. The same method of hexagon-doubling was also apparently used by al-Kashi since he used a polygon with 3×2^{28} sides.

The question remains: what was it that enabled Liu Hui to carry out this computation to a 3072-gon? Was it sheer labour, or did Liu Hui have access to a more efficient technique of square-root extraction than was available to Archimedes? We do know that early Chinese techniques of square-root extraction were geometric like those of Ptolemy illustrated by Theon of Alexandria. But most likely Liu used a technique similar to what is today called Horner's method. We do not know whether the techniques of square-root extraction used to calculate the value of $\sqrt{2}$ in the *sulba sutra* were based on a method of successive numerical approximations, or on a geometric technique. My own reading strongly favours successive numerical approximations. Thus, the fact that an efficient numerical algorithm for extracting square roots using the decimal place-value system appears for the first time in the *Aryabhāṭīya* is no reason to assume that this was the first time it was formulated. The date of the Bakshali manuscript seems to me unimportant in this connection, and the detailed physical description of it quite irrelevant to the question of fixing its date,⁶⁰ which could be better settled using radiometric rather than palaeographic techniques.

In any case, Aryabhata proceeds to a finite difference technique, and the *Yuktibhāṣā* commentary explains that this enables the preceding computation of the circumference of a circle to be carried out without having to extract square roots. It is not possible here to go into all the intricacies of this method, which requires a separate article by itself. The method, in outline, is as follows. One divides the circumference into a number of unequal small parts by dividing the east-west side of the square into a number of equal parts (Fig. 7). The problem now is to calculate the lengths of the arcs and to sum them up, as the number of parts is made very large. This calculation is carried out by an iterative process⁶¹ of approximation, which leads to the infinite series:

$$\frac{\text{Circumference}}{8} = r \left(1 - \frac{1}{4} + \frac{1}{5} - \frac{1}{7} + \dots \right),$$

corresponding to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

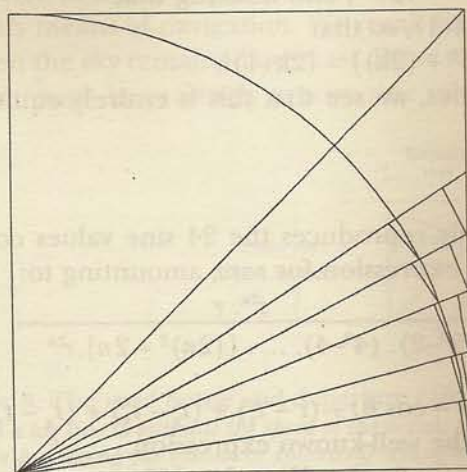


FIG 7. Aryabhata's technique of computing arcs and sines

Dividing the side of the square into a number of equal parts (which could be done) divided the circle into a number of unequal arcs. The figure shows the right-angled triangles used to estimate the lengths of these arcs.

Though this series converges slowly, its convergence was greatly accelerated by precise estimates of the 'error term' obtained after n terms of the series.

Using the geometric interpretation of sines, it is not difficult now to see how the finite difference technique initiated by Aryabhata led to the series expansions for sines and cosines stated by Madhava (1340–1425). The key passage is quoted by K.V. Sarma,⁶² and also by Bag⁶³ from the *Tantrasangraha*⁶⁴ (1501) of Neelkantha (1443–155) which gives an iterative procedure for the calculation. The passage may be re-translated as follows:

Multiply the arc by the square of the arc, and repeat [any number of times]. Divide by the product of the square of the radius times the square of successive even numbers increased by that number, [multiplication being repeated the same number of times]. Place the arc and the results so obtained one below the other and subtract each from the one above. These together give the *jiva*, collected here as found in the expression beginning with *vidvāns*, etc.

As is well known, Indian astronomy deals not directly with sines and cosines but with these quantities multiplied by the radius r of a standard circle. The *jivā* corresponds to $r \sin \theta$, while the *sāra* corresponds to $r(1 - \cos \theta)$. Let s denote the arc and t_n denote with n th expression obtained by applying the rule cited above. The rule requires us to calculate as follows. (1) Numerator: multiply the arc s by its square s^2 , this multiplication being repeated n times to obtain $s \cdot \prod_{k=1}^n s^2$. (2) Denominator: Multiply the square of the radius r for successive values of k by $[(2k)^2 + 2k]$, repeating this multiplication n times to obtain $\prod_{k=1}^n r^2 [(2k)^2 + 2k]$. Thus the n th iterate is obtained by:

$$t_n = \frac{s^{2n} \cdot s}{(2^2+2) \cdot (4^2+4) \cdot \dots \cdot [(2n)^2 + 2n] \cdot r^{2n}}$$

There is no ambiguity of interpretation here because the passage beginning with *vidvāns*, etc. is preserved in the *Aryabhatiyabhasya* of Neelkantha and the *Karanapaddhati*.⁶⁵ This interpretation correctly yields the five values of t_n listed there. The rule further says:

$$jivā = (s - t_1) + (t_2 - t_3) + (t_4 - t_5) + \dots$$

Substituting

$$(1) jivā = r \sin \theta,$$

$$(2) s = r\theta, \text{ so that } s^{2n+1}/r^{2n} = r\theta^{2n+1}, \text{ and noticing that}$$

$$(3) [(2k)^2 + 2k] = 2k \cdot (2k+1), \text{ so that}$$

$$(4) (2^2+2)(4^2+4)\dots[(2k)^2 + (2k)] = (2k+1)!,$$

and cancelling r from the sides, we see that this is entirely equivalent to the well-known expression:

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

This truncated series correctly reproduces the 24 sine values computed by Madhava. A similar rule gives an iterative expression for *sara*, amounting to:

$$t_n = \frac{s^{2n} \cdot r}{(2^2-2) \cdot (4^2-4) \cdot \dots \cdot [(2n)^2 - 2n] \cdot r^{2n}}$$

and,

$$r(1 - \cos \theta) + (r - t_1) + (t_2 - t_3) + (t_4 - t_5) + \dots,$$

which is again equivalent to the well-known expression:

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

That sine differences were proportional to cosines had already been noted long ago;

but derivatives and infinite series were not restricted to sine and cosine alone. One finds⁶⁶ derivatives of somewhat complicated expressions like $\sin^{-1}(p \sin w)$ or $p \sin w / (1 + p \cos w)$, together with a variety of other infinite series expansions.

In particular, a slight extension of the earlier infinite series expansion for π led to the infinite series expansion for arctan (YuktiBhāṣā, p.113)

$$\arctan \theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \dots$$

The above series as is well known can also be used to derive rapidly convergent expansions for π , and this 'Gregory's series' method was used in Europe from 1699 until the beginning of the twentieth century.

This technique clearly did not diffuse to China and Central Asia, where Al Kashi used the older technique to derive more precise values of π —that technique could not be generalized to derive more precise values of sines and cosines. It is possible that the advantages of the infinitesimal technique were not immediately obvious, and that mathematicians in Central Asia and in China (then under strong Central Asian influence) still preferred the older way of repeatedly extracting square roots. It is more likely that they simply did not know about this new technique since the speed with which innovations diffused was slow, so that the information would have diffused over a time scale much longer than what we take as 'normal' today. It was this diffusion that was prevented by the arrival of the Europeans, who not only systematically acquired manuscripts, quite likely even paying for them in many cases, but who also systematically disrupted the sea trade through a more straightforward variety of piracy.

5. ASTRONOMY

Early knowledge of astronomy has been painted in such ritualistic colours by Western historians of science that one tends to forget the earlier practical importance of astronomy for time keeping and navigation. Time keeping today means a digital watch; yesterday it meant the calendar. The calendar was not only a ritual calendar, but it was at the same time an agricultural almanac, deciding the good days for sowing and harvesting. It was also a secular calendar, having a separate civil year, which decided the days for holding markets or repaying loans, etc. Prior to the magnetic compass, presumably introduced by the Arabs, the stars were the only means of navigation. We recall Fa Hsian's tale of drifting at sea during a long period when the sky remained overcast. Prior to the chronometer, the way to fix longitude at sea was through increasingly precise sine values; this presumably was part

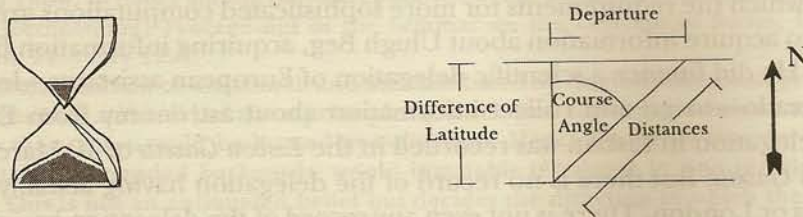


FIG 8. The sand bottle and departure calculations

The sand bottle was used with a log line to estimate the speed of the vessel, hence the distance travelled. This was used to compute the departure by solving the (plane or spherical) triangle on the right. The course angle was fixed using a magnetic or stellar compass. The difference in latitudes was measured, for example from the Pole Star altitudes by using finger measurements or instruments such as the Kamal. This method was more accurate than the European method of "dead reckoning", which relied upon the magnetic compass and charts, and only later brought in sine tables.

of the reason for the search for extremely precise values of π that we encountered above. This method of *tirfa* calculation is shown in Fig. 8.⁶⁷ Thus, astronomy was of central importance to both the economic and the cultural life.

The sketch of the mainline story of cultural exchanges here is as follows. The similarities between *nakshatra*, *hsius*, and *manazil* have been widely noticed. Siddhantic astronomy was systematically translated into Arabic, and came to be famous as the Sind Hind tradition. This process of translation continued for several centuries, but the translations were by no means passive—for example, Al-Bīrūnī explicitly commented on the divergence between the mythological explanation, to which the Siddhantic traditions pay obeisance, and the actual method of computation used in the Indian theory of eclipses. Again, the Arab-Islamic calendar has remained lunar, compared to the Siddhantic luni-solar system. A little earlier Indians had travelled to China and made a calendar there. The term *Qiyao* (seven luminaries) first appeared in China in 230 CE with the translation of the *Matang Sutra*.⁶⁸ Not only did Qutan Xida (Gautama Siddhartha, 718–29 CE) bring in the symbol for zero in the *Kaiyuan zhanjing*, which he edited, but Qutan Luo (665–98 CE), the first of the Gautama clan of experts in astronomy and calendar making, had also preceded him as the Director of the Astronomical Bureau in China.

Many standard Western histories of science add to this story the corollary that the entire Siddhantic tradition was derivative from Ptolemy earlier since 'Romaka', a plausible distortion of 'Rome', is stated by Varahamihira as one of the five Siddhantic traditions. (According to Aryabhata, Romaka is the point on the equator 90° west of Lanka, which is not the island mentioned in mythology but the point at which the meridian of Ujjayani meets the equator. Rome was, of course, known because of the extensive trade with the Romans.) We have already seen earlier that Ptolemy used only the chord as distinct from the sines used in the Indian tradition, and that (compared to Varahamihira) he did not even know a good way to extract square roots, relying, as he did, on the geometrical tradition as distinct from a numerical and algorithmic tradition. However, since this is not the issue here, I will only point out that it requires a complete lack of critical sense to imagine that Ptolemy, living in Egypt and having access to the library of Alexandria, represented the pure Greek tradition because he wrote in Greek, while Varahamihira also represented the same tradition since his use of the term 'Romaka' suggests that he had heard of Rome!

To return to the general theory of information flow and the theme of different kinds and stages of non-transmission of information, let us jump across to Jai Singh. The time gap between Ulugh Beg and Jai Singh gives some indication of the time scale of diffusion, and the slow rate at which the requirements for more sophisticated computations grew. While Jai Singh was able to acquire information about Ulugh Beg, acquiring information from Europe was not so easy. He did finance a scientific delegation of European assistants—led by a Jesuit priest Fr. Figuerado—to go and collect information about astronomy from Europe. The arrival of the delegation in Lisbon was recorded in the *Lisbon Gazette* of 10 March 1729, and also in the *Paris Gazette*. But there is no record of the delegation having actually visited centres such as Paris or London. There is not even any record of the delegation having visited the Coimbra University, then the only centre of higher learning in Portugal. The delegation returned in 1730 with some old and out-of-date information—it brought back La Hire's 1702 astronomical tables instead of the accurate tables of Flamsteed which Newton had used.⁶⁹ Other Europeans with whom Jai Singh was involved till 1733 also were unsympathetic to or did not seem to have heard of Newton or Kepler or even Galileo. Whatever the causes in this particular case, the net effect was information accumulation due to an unequal exchange of

information. Later, the internal information-sharing network was disrupted by European colonial intrusion in quite the same way as the internal trade network.

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- 1 Samuel P. Huntington, *The Clash of Civilizations and the Remaking of the World Order*, Viking Press, New Delhi, 1997.
- 2 John North, *The Fontana History of Astronomy and Cosmology*, HarperCollins, London, 1994.
- 3 Martin Bernal, *Black Athena: The Afroasiatic Roots of Classical Civilization, Vol. 1: The Fabrication of Ancient Greece 1785–1985*, Vintage, London, 1991.
- 4 'The sole trigonometric function used by Ptolemy is the chord', G.J. Toomer, *Ptolemy's Almagest*, Duckworth, London, 1984, p. 7.
- 5 J. Itard, 'Pure and Applied Mathematics,' in Rene Taton (ed), *Ancient and Medieval Science, Part II, Science in the Greco-Roman World*, Thames and Hudson, London, 1963, p. 297.
- 6 As Max Weber pointed out, people by and large only want to continue living the way to which they are accustomed, so a change in the lifestyle provides a strong motive for aggression. Max Weber, *The Protestant Ethic and the Spirit of Capitalism*, George Allen and Unwin, London, 1968, pp. 60–61. The context of Weber's observations was the empirically observed difference between an industrial society motivated by the goal of 'profit maximization' and an agricultural society.
- 7 This assertion goes well beyond what Toynbee has called 'barbarian incursions'.
- 8 To put it more formally, we have no way to estimate the a priori probability of a relation between the stars selected at random. The differences in the estimate of this probability, between H. Arp and his opponents, span 200 orders of magnitude.
- 9 For a nice photograph of this elephant and its *mahavat* and his *ankus*, see the frontispiece in D.A. Mackenzie, *Pre-Columbian Mythology*, Gresham Publishing Co., London.
- 10 The commerce between India and Egypt is known to have involved Indian elephants. Polybius recorded that in the battle of Raphia between Ptolemy and Antiochus, most of Ptolemy's elephants 'declined to combat, as is the habit of African elephants, for being unable to stand the smell and the trumpeting of the Indian elephants (of which he placed 60 under the command of his foster-brother Philip) and terrified, I suppose, also by their great size and strength they at once turned tail and took to flight before they got near them.' Polybius, *The History*, III, Bk V, 84, 6–84, 7, p. 205, cited by R.N. Saletore, *Early Indian Economic History*, popular Prakashan, Bombay, second ed, 1993, p. 208.
- 11 25 November 1915, 16 December 1915, 27 January 1916.
- 12 M.D. Srinivas, 'The Methodology of Indian Mathematics and its Contemporary Relevance,' *PPST Bulletin*, 12 (1987) 1–35.
- 13 Jean Filliozat, 'Ancient Indian Science', in Taton (ed), *Ibid.*, p. 151.
- 14 George Gheverghese Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics*, Penguin Books, 1992, p. 4.
- 15 So far as I know, to date I am the only exception! See C.K. Raju, 'The Mathematical Epistemology of Sunya', (forthcoming) Proceedings of the Seminar on the Concept of Sunya, INSA and IGNCA, New Delhi, February 1997.
- 16 The particular relation of the church and the state based on Augustinian theology required not only an eternal heaven and hell but also a principle by which God could decide whom to send where—the principle that causes could be located in individuals. Hence the belief that, for any innovation, a causal analysis, extended backwards, would invariably terminate in one individual who had the priority. This is not an antiquated belief but decides the direction in which the Dunkel Draft has required us to modify our patent laws. More details may be found in the chapter, 'The curse on "cyclic" time' in my forthcoming book, *The Eleven Pictures of Time*.
- 17 Martin Bernal, *ibid.*
- 18 f. 189, cited in V.N. Sharma, 'Astronomical efforts of Sawai Jai Singh—A Review.' In: *History of Oriental Astronomy* (Proc. IAU Colloquium No. 91, 1985), G. Swarup, A.K. Bag, and K.S. Shukla (eds), Cambridge University Press, 1987.
- 19 *Ibid.*
- 20 Vol. 2, pp. 15–6; Vol. 3, p. 24 (Jarett's edition).

- 21 Needham, *ibid.*, p. 43.
- 22 Richard S. Westfall, *Never at Rest: A Biography of Isaac Newton*, Cambridge University Press, 1983. Newton's earlier biographer, Charles Brewster, knew of Newton's theological obsessions, which filled the better part of Newton's life and works, but Brewster deliberately suppressed a cartload of Newton's papers, and then stated that Newton should be regarded as a religiously orthodox person in the absence of evidence to the contrary!
- 23 K. Marar and C.T. Rajagopal, 'On the Hindu Quadrature of the Circle', *Journal of the Bombay Branch of the Royal Asiatic Society*, 20 (1944) 56-77; C.T. Rajagopal and A. Venkataraman, 'The Sine and Cosine power series in Hindu Mathematics', with an addendum by K.M. George, *J. Asiatic Society of Bengal*, 3rd Series, 25 (1949) 1-13. C.T. Rajagopal, 'A Neglected Chapter of Hindu Mathematics', *Scripta Mathematica*, 15 (1949) 201-9; C.T. Rajagopal and T.V.V. Aiyar, 'On the Hindu Proof of Gregory's Series', *Scripta Mathematica*, 17 (1951) 65-74; C.T. Rajagopal and M.S. Rangachari, 'On an untapped source of medieval Keralese mathematics', *Archive for History of Exact Sciences*, 18 (1978) 89-102; C.T. Rajagopal and M.S. Rangachari, 'On medieval Keralese mathematics', *Archive for History of Exact Sciences*, 35 (1986) 91-9; K.V. Sarma, *A History of the Kerala School of Hindu Astronomy (in perspective)*, Vishveshvaranand Institute, Hoshiarpur, 1972. A more recent review is in A.K. Bag, *Mathematics in Ancient and Medieval India*, Chaukhamba, New Delhi, 1979. A popular account is in George Gheverghese Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics*, Penguin, 1991.
- 24 They had a special interest in learning Malayalam because the ancient Syrian Christian church in India used this language. It must also have been easy for the Jesuits to learn Malayalam because some Orthodox Syrian Christians also knew Greek, and the Mapilas used Arabic-Malayalam, so that some of them at least knew both Arabic and Malayalam. The Malayalam that the Jesuits learnt is exactly the same Malayalam in which Jyesthadeva's *Yuktibhasa* was written.
- 25 There is, of course, the early 1832 paper of Charles M. Whish, 'On the Hindu quadrature of the circle, and the infinite series of the proportion of the circumference to the diameter exhibited in the four Sastras, the *Tantrasamgraham*, *Yukti-bhasa*, *Carana Padhati* and *Sadratnamala*, Tr. Royal Asiatic Society of Great Britain and Ireland, 3 (1835) 509-23.
- 26 See the note on the church-state relationship and Augustine's theology above, relating this to the principle of priority.
- 27 For a short account of this quarrel, see the last but one paragraph of Stephen Hawking, *A Brief History of Time*, Bantam, New York, 1988.
- 28 For example, *Rahmani* of Kunhi Kunhi Koya of Kavaratti, Lakshadweep (to be published in English translation) which contains tables from a British sailing manual.
- 29 C.K. Raju, 'Kamal or Rapalagai', preprint, giving details of a field visit.
- 30 R.N. Saletore, *Early Indian Economic History*, Popular Prakashan, Bombay, 1993. Xinru Liu, *Ancient India and Ancient China: Trade and Religious Exchanges, AD 1-600*, Oxford University Press, Delhi, 1994.
- 31 Pliny, *Natural History*, II, Bk vi, Ch. 16, p. 63; cited in R.N. Saletore, *ibid.*, p. 88.
- 32 Not Ibn Majid. This information is quite self-consistent, for 'Malemo' clearly comes from Mualim/Malmi (=Navigator), and Cana/Kanha is a common name on the Diu coast, perhaps related to the port of Dwarka. At any rate, Vasco da Gama, who did not know what Guzerat was, could not have been expected to have concocted this, though no doubt later Portuguese historians had ample motivation to concoct the story that it was the legendary Ibn Majid who took Vasco da Gama across.
- 33 The admiral Nearchus whose voyage is described in Arrian, *Indika*, cited by Saletore, p. 296.
- 34 Some confusion has been caused by Kautilya's use of the terms Ashadha, and Kartika as the proper times for sailing out and in. By the time of the *Periplus*, it seems to have been an established procedure to take advantage of the monsoon winds by sailing out from Egypt around July, and sailing back around October-November. It has been suggested that Kautilya must have meant that people must sail out from foreign lands in Ashadha (June-July), and sail to foreign lands in Kartika (October-November). There seems to me, however, another possible explanation. The *Vedanga Jyotisha* calendar was falling out of phase with the observations until it had to be completely abandoned by the time of Varahamihira. Hence, the correlation between the luni-solar and solar calendar need not have been the same as Kautilya's time, and Kautilya's use of the term Ashadha need not refer to the same season as we mean now. Kautilya, *Arthaśāstra*, Bk II, Ch. XXVIII, p. 142. Saletore, *ibid.*, pp. 281-284.
- 35 Fa Hian, *Record*, pp. 113-14, cited in Saletore, pp. 528-29.
- 36 Aryabhata states that 'the globe of the Earth stands (supportless) in space... and is spherical. Just as the bulb of a Kadamba flower is covered all around by blossoms, just so is the globe of the Earth surrounded by all creatures, terrestrial as well as aquatic....Just as a man in a boat moving forward sees the stationary

- objects (on either side of the river) as moving backward, just so are the stationary stars seen by people at Lanka (on the equator) as moving exactly towards the west.' *Gola*, 6-9; *Aryabhata of Aryabhata*, (eds and trs) K.S. Shukla and K.V. Sarma, INSA, New Delhi, 1976, pp. 118-119.
- 37 Varahamihira accepts, of course, the roundness of the earth, but feels that it is the celestial sphere which rotates, and not the earth. His argument is that if the earth rotated, there would be a wind on the surface of the earth. Not finding this argument quite convincing, and relying on Aryabhata's river analogy, he adds that eagles which fly very high would be unable to return to their nests. An important medieval instrument was the astrolabe, which embodied a planispheric projection of the cosmic globe. The idea of a spherical earth was already incorporated in the Siddhantic literature translated into Arabic amongst others by Al-Birūnī himself who translated the *Pancsiddhantika* of Varahamihira. G. Thibaut and Sudhakara Dwivedi, (trs), *Pancsiddhantika*, reprint, Chowkhamba Sanskrit Series, Varanasi, 1968, 13.27 and 13.9-13, p. 72, p. 70.
- 38 Al-Birūnī's method was, first, to measure the height of a hill by measuring the angles subtended by the hill at two points a known distance apart, and applying the usual trigonometric formula. He then climbed the hill and measured the angle of the dip of the horizon. Assuming the earth to be a perfect sphere, the line of sight is tangential to the sphere, hence orthogonal to the radius, and a simple calculation gives the radius, hence the circumference of the earth. Al-Birūnī's value of the radius of the Earth was equal to 3938.77 English miles (using 1 Arabic mile = 1.225947 English miles). This compares favourably with the mean radius of the curvature of the reference ellipsoid, at his latitude, which is 3947.80 miles. Al-Birūnī's self-constructed instrument could accurately measure angles to at the most a minute of the arc, so the key to the accuracy is the high-precision finite-difference interpolation technique used to construct sine tables. Al-Birūnī translated the *Karana Tilak* of one Vijaya Nandi of Benares, for Zij calculations, as the *Ghurrat-uz-Zijat*, the manuscript of which was discovered in 1959. The *Karana Tilak* falls in between the *Karana Khandakhadya* of the sixth-century Brahmagupta (who explicitly uses the idea that the first sine differences are proportional to cosines, and whose work Al-Birūnī had thoroughly studied) and the *Karana Kutuhula* of the second Bhaskara. See S.S.H. Rizvi, 'A Newly Discovered Book of Al-Birūnī: 'Ghurrat-uz-Zijat', and Al-Birūnī's 'Measurements of Earth's Dimensions'. In: H.M. Said (ed) *Al-Birūnī Commemorative Volume*, Hamdard Academy, Karachi, 1979, pp. 605-80.
- 39 G.H. Tibbetts, *Arab navigation in the Indian Ocean before the coming of the Portuguese* (Tr. of Ibn Majid's *Fawa'id*), Royal Asiatic Society, London, 1971.
- 40 There were two types of zams in Arabic navigation. There was another definition of the zam as the time for one of the eight watches in a day, implying that the zam referred to a duration of three hours. This zam, translated into distance, led to some confusion about the representations of distances. This particular definition of the zam is one I have ascertained at first hand, from the Lakshadweep Islanders. The historical sequence of this definition in the context of Arabic traditions of navigation is not clear to me. However, it seems clear that the Arabic zam is derived from the Sanskrit 'Yama', one of the oldest time units in India which continues to be used as 'prahara' today as in dopahara = afternoon.
- 41 Despite the heroic efforts of the Western historians of science, it is not so clear that there was any actual person named Euclid or that there was a unique text in existence from which all later texts are derived. The only Euclid known to the early Greek tradition is Euclid of Megara. For many centuries, medieval Europeans identified Euclid of Megara as the author of the *Elements*, until the identification was given up as hopeless. Only a later Greek source, the fifth-century Proclus, believed that Euclid was a real person who had lived perhaps seven centuries earlier.
- 42 T.L. Heath, *The Thirteen Books of Euclid's Elements*, Dover, New York, vol. I, p. 326.
- 43 Heath, *ibid.*, p. 349.
- 44 Two sets of quotation marks are needed because of the following. (1) The 'Pythagorean theorem' did not, of course originate with Pythagoras, Pythagoras' claim to it rests on some clearly borrowed Egyptian knowledge of the triples of numbers, and the statement of Proclus that 'rumour' credits Pythagoras (some eight hundred years before Proclus) with having found a proof. (2) It now emerges that there is no proof of it even in the *Elements*, so the proposition in question cannot be called a 'theorem' until we have settled on a formal theory which reinterprets the *Elements*, and within which this is a theorem.
- 45 There is something to be said for the possible ritualistic elements in the *Elements*. Some ritualistic elements, such as the last sentence being a repetition of the first, have long been noticed. This probably is a metaphor for some kind of cyclicity of time which was an important aspect of Egyptian and Neoplatonic beliefs. Geometry, as the name suggests, probably originated in the problem of measuring land areas. And

- the rectangle is the most natural figure for which the area is defined. Thus, the non-definition of area in the *Elements*, together with the emphasis on triangles instead of rectangles, suggests a ritual basis to the geometry of the *Elements*: triangles, as depictions of female genitalia, were used in Egypt as fertility symbols, related to agricultural production. D.P. Chattopadhyaya, *Lokayata*, ICPR, New Delhi.
- 46 The Epicureans, according to Proclus, maintained that any ass knew of Euclid's 20th Proposition that two sides of a triangle are together greater than the third, since the ass went straight towards the hay, and did not pursue two sides of a triangle by first going in one direction and then changing direction.
- 46a More precisely, one should distinguish between a metric approach which involves actual measurements, hence empirical procedures, from a formal metric approach, such as Birkhoff's, which formally introduces a notion of metric. Subsequent references to 'metric' in this article, in the context of traditional geometry, mean 'metric-empirical' rather than 'metric-formal'. This sort of precision in the text might, however, make the text unreadable for the audience it addresses.
- 47 Heath, *ibid.*, p. 364.
- 48 Heath, *ibid.*, p. 181.
- 49 In Plato's time, Greek notions of 'equality' were of a much more down-to-earth kind, for that democracy bypassed slaves and women who made up more than 80 per cent of the population—Plato was more of a republican than a democrat! The later Neoplatonic doctrines of Plotinus were more in tune with the kind of stress on 'equality' that we find in the *Elements*, and the influence of Neoplatonic theology on the Mutazilah is well known.
- 50 C.K. Raju, 'The Mathematical Epistemology of Sunya', forthcoming, in Proceedings of the *Seminar on the Concept of Sunya*, INSA and IGNCA, New Delhi, 12–14 February 1997.
- 51 The scheme of establishing equity by deducing geometrical truths from it may have originally had a political force to it that ought not to be underrated even if the scheme seems impractical today.
- 52 *Aryabhataiya of Aryabhata*, (eds and trs) K.S. Shukla and K.V. Sarma, INSA, New Delhi, 1976, pp. 38–45.
- 53 Joseph Needham, *The Shorter Science and Civilization in China, Vol. 2*, an abridgement by Colin A. Ronan of the original text, Cambridge University Press, 1981, p. 43.
- 54 Needham, *op. cit.*
- 55 *Aryabhataiya Bhashya, Ganitapada*, p. 42. Text cited in full in T.A. Sarasvati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarasidass, New Delhi, 1979, p. 157.
- 56 A.K. Bag, 'Madhava's sine and cosine series', *Indian Journal of History of Science*, 11 (1) 54–57, 1976. Bag quotes a key passage from Neelkantha's *Tantrasamgraha*, S.K. Pillai (ed), Trivandrum Sanskrit Series 188, Trivandrum, 1958, p. 19. Chapter 2, Verse 7 gives the simplified value of $355/113$. The accuracy is actually to ten decimal places, and not eleven as stated by Bag.
- 57 K.V. Sarma (tr), *The Ganitayuktibhasa of Jyeshthadeva*, unpublished.
- 58 Neelkantha Somayaji, *Aryabhataiya Bhashya*, text quoted in T.A. Sarasvati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarasidass, Delhi, 1979, p. 155. (Tr. modified by the author). Also quoted in S. Grover, *History of Development of Mathematics in India*, Atma Ram and Sons, Delhi, 1994, p. 211.
- 59 Ho Peng Yoke, *An Introduction to Science and Civilization in China*, Oxford University Press, New Delhi, 1985, pp. 74–76.
- 60 Takao Hayashi, *The Bakhshali Manuscript*, Egbert Forsten, Groningen, 1995.
- 61 A summary description of this process may be found in T.A. Sarasvati Amma, *ibid.*, pp. 159–167.
- 62 K.V. Sarma, *A History of the Kerala School of Hindu Astronomy (in perspective)*, Vishveshvaranand Institute, Hoshiarpur, 1972, p. 16.
- 63 A.K. Bag, 'Madhava's sine and cosine series,' *Indian Journal of History of Science*, 11 (1) 54–57, 1976.
- 64 *Tantrasamgraha*, S.K. Pillai (ed), Trivandrum Sanskrit Series 188, Trivandrum, 1958, p. 19.
- 65 Trivandrum Sanskrit Series 126, Chapter 6, verses 14–15.
- 66 *Tantrasamgraha* V-53–54, and *Sphutanirnaya* III. 19–20, as pointed out by K.V. Sarma (ed and tr) *Ganitayuktibhasa of Jyeshthadeva*, pp. 33–34 (unpublished).
- 67 Taken from my paper, 'Kamal or Rapalagai,' cited earlier.
- 68 The seven luminaries were the sun, the moon, and the five planets. The calendar system during the Tang period also included the Jiuzhali—the Nine Controllers Calendar—which included the two imaginary invisible planets used in computations by Indian astronomers—Rahu and Ketu. Yoke, *ibid.*, Chapter IV.
- 69 The Jesuits had carried the telescope to China in 1618, barely two years after Galileo announced his telescope discoveries, and the telescope is described in Chinese books from 1615 onwards. But after the condemnation of Galileo in 1632, they seem to have neglected the heliocentric theory, as is evident from its decline in Chinese books from 1635 onwards. Needham, *ibid.*, p. 216.