

A new way to teach calculus based on a new history and philosophy of math

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It is believed that calculus was invented by Newton and Leibniz, and the correct way to teach it is to use limits, using Dedekind's formalisation of real numbers, constructed using axiomatic set theory, based on the formalist philosophy of Russell and Hilbert which corrected the deductive method of Euclid's *Elements*.

The historical claim follows the stock pattern of Western history which attributes all achievements first to Hellenic culture (in this case, “Euclid”, Archimedes, etc.), and then to post-renaissance Europe (in this case, Newton, Leibniz, etc.). Others are typically erased. The links of this history to racism are manifest from the foolish arguments of Hume, Kant,¹ and other leading Western philosophers, who justified this erasure by claiming that only white-skinned people were creative. This agenda was furthered by making “Euclid's” purported axiomatic-deductive method philosophically *normative* to declare other ways of doing things as not quite mathematics—as “land-surveying” not geometry, for example.² Thus, today, we have the “philosophical” belief that the only right way to do mathematics is the Western way which focuses on rigor through metaphysical deductive proof. Apart from a racist bias, this involves a *religious* bias,³ as follows.

First, this way to do mathematics supposedly originated with a “Euclid” for whose existence there is nil evidence.⁴ As for the *Elements*, its first proposition involves an empirical proof as does the key proposition 4 known as the SAS theorem. These empirical proofs were eliminated only at the turn of the 20th c. by Russell and Hilbert who disregarded these *facts*, and just went along with the *myth* of Euclid and his purported intention of metaphysical deductive proof, and hence turned SAS into a postulate.⁵ However, the resulting formalisations do not fit the *Elements*. Hilbert's synthetic axiom set, which reinterprets “equal” as “congruent”, clearly fails beyond proposition 35 which uses the term “equal” for incongruent areas. Birkhoff's metric axiom set makes the book *Elements*, itself, pointless, since the “Pythagorean theorem” can be proved in a single step (as in empirical Indian proofs) and does not need 46 intermediate propositions.⁶ In fact, the *Elements* was written at another time (+5th c. CE) by a different person (Hypatia) with a different philosophy of mathesis which philosophy persisted from Plato⁷ to Proclus,⁸ and originates from the sacred geometry of Egypt.⁹ This involved a belief in the soul and its past lives.¹⁰ Though part of original Christianity,¹¹ those beliefs (and the related philosophy) were cursed by the post-Nicene church,¹² and became taboo in the West thereafter. Post-Crusade rational theology reinterpreted Christianity as a doctrine of reason, by adapting the Islamic theology of reason (*aql-i-kalam*), taking into account its critique by al Ghazali.¹³ Theologians contended that God was bound by logic but free to create the empirical facts of his choice; hence they elevated metaphysics above empirical facts. The belief in metaphysical deduction as certain and universal actually originated with church doctrine, not the *Elements*.

Now, as a matter of commonsense, metaphysics can never be universal.¹⁴ The belief that metaphysical (deductive) proof provides greater certainty than empirical observation is peculiar to post-Crusade Christian rational theology. Not only is it against mathesis, it rejects *all* Indian systems of philosophy at one stroke, for the one and only one thing they all accept in common is the empirically manifest (*pratyaksa*) as a sure means of proof. In fact, mathematical theorems would change with changes in logic, and logic is neither culturally universal (e.g. Buddhist logic of *catuskoti*, Jain logic of *syadvada*)¹⁵ nor empirically certain (e.g. quantum logic,¹⁶ or the logic of natural language), so the nature of logic underlying deduction *must* be decided empirically, to avoid a bias. This bias extends also to the postulates which are, in principle, arbitrary for a formal theory. But, for example, students are today taught that calculus requires the continuum, which is contrary to Islamic and Indian beliefs in atomicity. Further, all practical applications of calculus today use a computer which uses a discrete number system and a floating point arithmetic, different from the arithmetic of formal reals.¹⁷ This too is declared erroneous. Thus teaching formal mathematics inculcates a pro-Christian bias, and declares non-Christian and practical ways of doing math as wrong.

These racist and religious biases inherent in the church education system were globalised by colonialism,^{18,19} and are reflected in the way calculus is taught today. Limits are of no practical value and are supposedly done for the sake of rigor. In fact, however, engineering and science students of calculus are not actually taught formal limits, or formal reals or axiomatic set theory. They are only *told* about the intention of rigor, but *taught* to accept it on faith. At an advanced level, students do learn about formal limits, and formal real numbers, but neglect formal set theory. They remain unfamiliar with the paradoxes of set theory which persist even after its axiomatisation (such as the Banach-Tarski paradox), and unaware that set theory can be used to “rigorously” derive conclusions (that one ball of gold can be turned into ten) laughable from a practical perspective. What value such rigor if rigor it indeed be? These biases are supported by social control: the formal mathematics community rigorously excludes any forum, whether in the classroom or outside, where such questions about mathematics can be raised and discussed.

Present-day calculus lacks conceptual clarity even about the definition of the derivative. Thus, defining the derivative as a limit forces a differentiable function to be continuous. However, in physics it is routinely necessary to differentiate discontinuous functions. The Schwartz derivative (which needs the Lebesgue integral) permits this, but it does *not* generalize the classical definition.²⁰ Now, science must be refutable, so we cannot switch definitions as convenient, but must stick to one definition. But neither definition of derivative can be used either in classical physics (where the differential equations are nonlinear²¹) or in quantum field theory²²—in both cases we need to differentiate and multiply. With the derivative as limit we cannot differentiate a discontinuous function, with the Schwartz theory we cannot multiply distributions.²³

So, how should calculus be taught? First, it is clear that if there is no universal notion of rigor, calculus ought not to be taught for “rigor”. On the contrary, now that the religious bias in the Western notion of rigor has been clearly pointed out, it would be unconstitutional in many countries to continue to teach calculus (or math) that way. For schools and undergraduate course, it should only be taught for its practical applications. These typically involve the numerical solution of various sorts of differential equations, today obtained using digital computers.

This was, in fact, the way the calculus actually originated in India.²⁴ Thus, the two key means of wealth in India were agriculture and overseas trade.²⁵ Indian monsoon-driven agriculture requires an accurate calendar,²⁶ which, in turn, needs accurate astronomical models. Likewise, the vast Indian overseas trade needed accurate navigation. Both required accurate trigonometric values. These (the finite

differences, actually) were calculated in a novel way by Aryabhata²⁷ in the 5th c. by numerically solving the ordinary differential equations for sine and cosine by what is today called “Euler's” method.²⁸ (Euler, of course, studied the Indian calendar and wrote on it.)

Aryabhata's elegant numerical technique easily led to 24 trigonometric values precise to 5 decimal places over the entire arc. Over the next thousand year, this technique was developed by mathematicians and astronomers of the Aryabhata school in Kerala (wrongly called the Kerala school), who used infinite series expansions to obtain trigonometric values precise to 9 decimal places.²⁹

Precise trigonometric values were then badly required for the solution of the European navigation problem (specific to Europeans who used maps and lacked a technique of celestial navigation).³⁰ This was the major European scientific problem of the time, and large prizes were offered for its solution by various European governments. The key difficulties then were to determine latitude, longitude and loxodromes.³¹ Navigational theorists like Simon Stevin used Aryabhata's values from Arab sources.³² Later, more accurate values were transmitted direct from India in the 16th c. by Jesuits such as Matteo Ricci,³³ and others in the Jesuit college located in Cochin, in the vicinity of the Aryabhata school in Kerala. These Jesuits translated Indian texts, in Toledo mode, and sent them back to Europe. Those were first used by Mercator, then Christoph Clavius,³⁴ Tycho Brahe and his assistant Kepler, they went on to Galileo, and his student Cavallieri, and Fermat and Pascal,³⁵ and then diffused throughout Europe.

Europeans then (and now) failed to understand how Indians summed the infinite series. Because they mixed math with religious beliefs, they insisted that math must be perfect. Hence, Descartes³⁶ stated that summing an infinite series, to obtain the number π , was beyond the human mind. Thus, physically summing the entire series would take an eternity of time, while stopping at any point would mean an approximation, which was not perfect, hence not mathematics. Galileo concurred, in his letters to Cavalieri. Newton thought³⁷ he had resolved the difficulty with fluxions (which only made time metaphysical,³⁸ and hence led to the failure of his physics³⁹). Berkeley attacked Newton,⁴⁰ and the story from that point to the present-day is well known.

Now, a formal description of the Indian technique of summing infinite series would be that they did order counting⁴¹ by discarding infinitesimals in the field of rational functions, *larger* than formal reals. (Rational functions were treated as analogous to rational numbers.) But, what is today called the “Archimedean property” fails in a such a field, so that limits are not unique, and infinitesimals must be discarded. Note that we are speaking of an actual field extension of formal reals, not an intermediate⁴² non-standard extension. Naturally, Indians did not suppose like Europeans that there was some sort of God-given notion of infinitesimal. In fact, this “discarding of infinitesimals” involved a sophisticated philosophy which I have called zeroism.

Zeroism takes off from Buddhist *sunyavada*, but for its practical benefits, and *without* reference to any Buddhist texts, like those of Nagarjuna.⁴³ It is a realistic philosophy which regards idealism as *erroneous*. In particular, the idea of mathematics as perfect, which is at the root of European epistemological difficulties with the calculus, stands rejected.

The issue of discarding small (“infinitesimal”) quantities (as in summing the infinite series for π , or in the frequentist interpretation of probability) now appears as a problem of *representation*, which is readily resolved. Incidentally, zeroism also resolves the problem of the frequentist interpretation,⁴⁴ which cannot be resolved by conventional limits.

Calculus has been successfully taught with this new philosophy over the last two years, in teaching experiments conducted in universities in India and Malaysia.⁴⁵ Eliminating the theology from math makes math easy. This has been demonstrated by teaching calculus very quickly (typically in 5 days) even for non-math students (who did not do any advanced math in school). The post-test included a randomized selection of problems from published calculus question banks. Other demonstrated advantages of the new pedagogy of the calculus include the ability of school students to do advanced modelling (such as the amplitude-dependent motion of the simple pendulum, typically done through Jacobian elliptic functions).⁴⁶

Reports of the new history and philosophy of math, and these pedagogical experiments have already been widely published in scholarly articles, books,⁴⁷ blogs,⁴⁸ and media.⁴⁹ This paper aims to bring home these ideas directly to ICME participants.

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- calculus from India to Europe in the 16th c. CE*, Pearson Longman, 2007. For a quick summary, see Ghadar Jari Hai, 2(1) (2007) pp. 26–29. Full text: <http://IndianCalculus.info/Cultural-Foundations-Mathematics-review-GJH.pdf>.
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- 28 For details, see C. K. Raju, “Infinite series and π ”, chp. 3 in *Cultural Foundations of Mathematics*, cited above.
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- 34 Christophori Clavii Bambergensis, *Tabulae Sinuum, Tangentium et Secantium ad partes radij 10,000,000. . .*, Ioannis Albini, 1607. As the title suggests, this table concerns not trigonometric values, as today understood, but values of functions such as the Rsine, which are what are given in Indian manuscripts. Stevin follows the same practice for his secant tables, *The Haven Finding Art*, cited above, p. 483.
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- 47 Apart from the books cited above, this includes *Making math easy* (submitted for publication).
- 48 See, for example, “Decolonising our Universities: Time for Change”, <http://globalhighered.wordpress.com/2011/09/11/decolonising-our-universities-time-for-change/>. This and other links are available also at this author's personal blog <http://ckraju.net/blog>. This includes a fairly extensive archive of the public

debate on decolonisation, as well as the issues earlier taken up with the National Knowledge Commission in India.

49 Media reports include the 2-page *New Strait Times* report on “Mathematics Made Easy”: http://www.nst.com.my/nst/articles/ISSUES_Mathematics_Noproblem/Article/, and the earlier 2-page report on “Reimagining the University”. News clips of this and other news reports together with the long (3-month) discussion in the press, on issues arising from the decolonisation conference, is archived at <http://ckraju.net/blog/?p=62>, and <http://ckraju.net/blog/?p=61>.