Hence, we obtain

$$n \cdot s_2(n) - s_3(n) = \sum_{j=1}^{n-1} s_2(j) \tag{3.114}$$

$$= \sum_{j=1}^{n-1} (j+1)s_1(j) - \sum_{j=1}^{n-1}\sum_{k=1}^{j} s_1(k) \tag{3.115}$$

$$= \frac{1}{2}s_3(n-1) + s_2(n-1) + \frac{1}{2}s_1(n-1) - \sum_{j=1}^{n-1}\sum_{k=1}^{j}\sum_{i=1}^{k} i. \tag{3.116}$$

The last term on the right is just the *vārasaṅkalitā*, while the remaining terms are all known except for s_3 which can hence be evaluated. The actual evaluation is a now a simple but tedious matter of elementary algebra. But the tedium is considerably reduced if we retain only the leading order terms, and obtain

$$\frac{s_3}{n^4} = \frac{1}{4} \qquad (n \text{ sufficiently large}), \tag{3.117}$$

in the precise sense that the remaining terms are numerically non-representable or insignificant, or infinitesimal for n infinite (or as large as is physically possible). The fourth-order *varga-varga saṅkalitā*, the fifth-order *varga-ghana saṅkalitā*, and higher-order series can be evaluated in a way similar to the *ghana-saṅkalitā*, using the result for the preceding *saṅkalitā*. Expressed in present-day terminology, the conclusion is that, for large n,

$$\frac{1}{n^{k+1}} \sum_{1}^{n} i^k \approx \frac{1}{k+1}, \qquad k = 1,\ 2,\ 3, \cdots. \tag{3.118}$$

The Results

Substituting the results (3.118) in the earlier expression (3.84), and remembering that $n\Delta r = r$, we finally get the value of the circumference,

$$\frac{\text{circumference}}{8} = r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \frac{r}{9} - \cdots. \tag{3.119}$$

The basic series is expressed through the *śloka*[81]

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादिविषमसंख्याभक्तमृणां स्वं प्रिथक् क्रमात् कुर्यात्॥ २.२७१॥

This may be translated as follows.[82]

> To the diameter multiplied by 4 alternately add and subtract in order the diameter multiplied by 4 and divided separately by the odd numbers 3, 5, etc.

This is described by the *Karaṇapaddhati* (VI, 1) as the accurate circumference. That is, if d is the diameter of the circle, then

$$\text{circumference} = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots . \tag{3.120}$$

This corresponds to the value of π given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{3.121}$$

This is the so-called Leibniz series. This series is *not* the best technique for calculating π, since the series (3.121) converges very slowly: some 10,000 terms are needed to obtain an accuracy of 4 decimal places. For an accuracy of four places after the decimal point, the above sum done on a computer needed to sum about 138,000 terms. Clearly, this sort of labour was impossible before digital computers, and, even with computers, one might have to pay some attention to the pile up of "rounding errors". (For 5 places after the decimals, a calculation done using double precision arithmetic is obviously good enough, since 10^5 floating point operations cannot propagate any "rounding errors" that far.)

This way of looking at things, however, overlooks some key points.

Deriving the Series Expansion for the Arctangent

First, once the idea was established, many other series expansions were obtained, and Whish has already recorded in 1832 a variety of fast-convergent expansions for π. In particular, Mādhava probably had obtained the series expansion for arctan, which involves only a slight extension of the above methods.

Referring back to Fig. 3.2, if Q' is any point on the arc PQ_n, and if OQ' is extended to meet the side square at P', then the *TantrasaṅgrahaVyākhyā/Yuktibhāṣā* states that an "equivalent argument" (*tulya nyāya*) shows that the arc PQ' is given by replacing r, in the above expression (3.119), by PP'. That is,

$$PQ' = PP' - \frac{PP'}{3r^2} + \frac{PP'}{5r^4} - \frac{PP'}{7r^6} + \cdots . \tag{3.122}$$

If the arc PQ' subtends an angle θ (= desired arc), and we use the notation $s = \text{Rsin}\ \theta$, $c = \text{Rcos}\ \theta$ (= *koṭijyā*), then we get from $PQ' = r\theta$, and $PP' = r\ \tan\theta = \frac{rs}{c}$, that

$$\text{arc}\ PQ' = \frac{rs}{c} - \frac{rs}{3c}\cdot\frac{s^2}{c^2} + \frac{rs}{5c}\cdot\frac{s^4}{c^4} - \frac{rs}{7c}\cdot\frac{s^6}{c^6} + \cdots . \tag{3.123}$$

This is expressed by the *śloka*[83] for "arcification" of the sine:

इष्टज्यात्रिज्ययोर्घातात् कोट्याप्तं प्रथमं फलं॥ २.२०६ ॥

ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम् ।
प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः ॥ २०७ ॥

एकत्र्याद्योजसङ्ख्याभिर्भक्तेष्वेतेष्वनुक्रमात् ।
ओजानां संयुतेस्त्यक्ते युग्मयोगे धनुर्भवेत् ॥ २०८ ॥

दोः कोट्योरल्पमेवेष्टं कल्पनीयमिह स्मृतम् ।

This may be translated:[84]

> The Rsine of the desired arc multiplied by the radius and divided by the Rcosine is the first result. Take the square of the Rsine as the multiplier, and the square of the Rcosine as the divisor, and multiply the first & etc. results to get the succeeding results. These are to be divided in order by the odd numbers, and the sum of the terms in even places is to be subtracted from the sum of the terms in the odd places. Remember to use the smaller of the two (Rsine and Rcosine) for this calculation.

It is clear that the expansion (3.123) is trivially equivalent to the more modern form

$$r\,\theta \;=\; r\,\tan\theta \;-\; \frac{r\,\tan^3\theta}{3} + \frac{r\,\tan^5\theta}{5} - \frac{r\,\tan^7\theta}{7} \;+\; \cdots\,, \qquad (3.124)$$

which, upon cancelling r, is the same as the "Gregory series" expansion for the arctan function:

$$\tan^{-1}\theta \;=\; \theta \;-\; \frac{\theta^3}{3} \;+\; \frac{\theta^5}{5} \;-\; \frac{\theta^7}{7} \;+\; \cdots\,. \qquad (3.125)$$

Deriving Rapidly Convergent Series for π

It is well known that the series (3.125) can be used to derive rapidly convergent expansions for π, using e.g. $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, so that

$$\frac{\pi}{6} \;=\; \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \;=\; \frac{1}{\sqrt{3}}\left\{1 \;-\; \frac{1}{3\cdot 3} \;+\; \frac{1}{5\cdot 3^2} \;-\; \frac{1}{7\cdot 3^3} \;+\; \cdots\right\}. \qquad (3.126)$$

This series requires only 9 terms for a precision of 4 decimal places. Small manipulations can be used to make the convergence even more rapid, and this was actually the way in which approximations to the value of π were calculated in Europe, by Sharp who in 1699 used "Gregory's" result to get 71 correct digits, by Machin who used a small improvement to get 100 correct digits, and whose method was used by de Lagny (1709, 112 digits), Vega (1789, 126 digits; 1799, 136 digits), Rutherford (1841, 152 digits; 1853, 440 digits), and Shanks (1873, 707 digits, of which 527 were correct). Indian mathematicians, however, being practical minded, computed π accurately to only the 11th decimal place, although 9 places