Name: ______________________

Student Number: ______________________

Course: ______________________

Age: ______________________

Date: ______________________

Math Group: Calculus without Limits
Exam, Pre-test: A

- Please attach this question paper and return it along with your answer sheet.
- This is not a competitive test. The aim is to obtain feedback to decide what to teach and how.
- Since the group is heterogeneous, you may find some questions too easy, or some may be too difficult. Attempt as many questions are you are able to.

1. (a) Define a complete metric space.

   (b) The least upper bound property for $\mathbb{R}$ says that if $A \subset \mathbb{R}$ is non-empty and bounded above, then $\exists \alpha \in \mathbb{R}$ such that $a \leq \alpha$, $\forall a \in A$, and if $a \leq b$, $\forall a \in A$ then $\alpha \leq b$. Assume the least upper bound property and prove that $\mathbb{R}$ is a complete metric space.

2. (a) Define "infinite set", "countable set", "uncountable set".

   (b) Prove that $\mathbb{R}$ is uncountable.

   (c) If $\mathbb{N}$ is the set of natural numbers, and $P(\mathbb{N})$ is its power set, does there exist a bijective map $f : P(\mathbb{N}) \to \mathbb{R}$?

3. (a) Write down the binary representation of 41.

   (b) Write down the binary representation of 2.5

   (c) Rewrite your answer using a mantissa between 1 and 2.

4. Given $g(x) = \begin{cases} x^2 - C, & \text{if } x < 4 \\ -\sqrt{C}x + 20, & \text{if } x \geq 4 \end{cases}$

   (a) Find the value of C which makes g continuous on $(-\infty, \infty)$.

   (b) With the above value of C, is g differentiable? Explain your answer.

5. (a) Suppose f_n is a sequence of Riemann integrable functions which converges to the function f on $(0, \infty)$, convergence being uniform on compact subsets. Is it true that f is Riemann integrable and that $\int_0^\infty f_n \to \int_0^\infty f$?

   (b) Suppose f_n is a sequence of differentiable functions which converges uniformly to the function f on $(0, 1)$. Is it true that f is differentiable and that the sequence of derivatives ${f_n}' \to f'$?

6. (a) Give an example of a real-valued function f which is not Riemann integrable on $[0, 1]$. Is this Lebesgue integrable?

   (b) Does there exist a Riemann integrable function on $(0, \infty)$ which is not Lebesgue integrable?

7. The following ten numbers were drawn at random from $[0, 1]$ using a uniform probability distribution: 0.23, 0.74, 0.18, 0.79, 0.51, 0.34, 0.67, 0.44, 0.11, 0.44.

   (a) Find the average.

   (b) Explain why it is not 0.5.

   (c) If the average does equal 0.5 at some stage, can subsequent draws of further random numbers change that value?

   (d) An unbiased coin is tossed 100 times. The first toss is tails, and the subsequent 99 tosses are heads. At the 101st toss (i) is the probability of tails greater than that of heads or (ii) is the probability of heads greater than that of tails?

8. Suppose a monkey is typing at random on a typewriter which has 50 keys (z and Z having been dropped), and suppose that the monkey is equally likely to strike any key.

   (a) What is the probability that the first six letters the monkey types will spell the word "Hamlet".

   (b) Suppose we consider the letters typed by the monkey in consecutive blocks of six letters. What is the probability p_n that the first n blocks of six letters will have the word "Hamlet"?

   (c) Does $\lim_{n\to\infty} p_n$ exist? If so, what is it?

9. Differentiate the following with respect to x

   (a) $\sin^n x . \sin nx$

   (b) $\sec^{-1} \dfrac{\sqrt{x}+1}{\sqrt{x}-1} + \sin^{-1} \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

   (c) $x - \log\left(2e^x + 1 + \sqrt{e^{2x} + 4e^x + 1}\right)$

10. Evaluate the following indefinite integrals.

   (a) $\int \sqrt{3x+2}\ dx$

   (b) $\int \log x\ dx$

   (c) $\int \dfrac{dx}{\sqrt{\sin^3 x \cdot \sin(x+\alpha)}}$