c. CE Mādhava of Saṅgamagrāma who first used them to derive a "table" of 24 very accurate trigonometric values. This table greatly improves upon the accuracy of a similar "table" of 24 values provided a thousand years earlier by Āryabhaṭa, and continuously improved upon since then by various people, including Bhāskara I and Vaṭeśvara. Where Āryabhaṭa's 24 sine values are accurate to the first sexagesimal minute, and Vaṭeśvara's 96 values are accurate to the second sexagesimal minute, Mādhava's 24 values are accurate to the third sexagesimal minute—about eight to nine decimal places.

II
THE SERIES EXPANSION FOR SINE AND COSINE

The "Taylor" series expansion for the sine is stated in a couple of verses, as follows.

> निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
> हरेत् समूल्युग्वर्गैस्त्रिज्यावर्गहतैः क्रमात् ॥ ४४० ॥
> चापं फलानि चाधोऽधो न्यस्योपर्युपरित्यजेत् ।
> जीवाप्त्यै, संग्रहोऽस्यैव विद्वान् इत्यदिना क्रितः ॥ ४४१ ॥

The key passage[12] may be translated as follows.[13]

> Multiply the arc by the square of the arc, and take the result of repeating that [any number of times]. Divide [each of the above numerators] by the squares of successive even numbers increased by that number [lit. the root] and multiplied by the square of the radius. Place the arc and the successive results so obtained one below the other, and subtract each from the one above. These together give the *jīvā*, as collected together in the verse beginning with *"vidvān"* etc.

Jīvā relates to the sine function. Etymologically, the term sine derives from *sinus* (= fold), a Latin translation of the Arabic *jaib* (fold for pocket, as in a shirt). What the Oxford English Dictionary does not mention is that *jaib* (= pocket) is a misreading of the Arabic term *jībā* (both terms were written as *jb*, omitting the vowels). Mathematically, however, as is well known, Indian mathematics and astronomy (like European mathematics in the 16th and 17th c. CE) dealt not directly with present-day sines and cosines but with these quantities multiplied by the radius r of a standard circle. Thus, *jīvā* (earlier *jyā*) corresponds to $r \sin\theta$, and is sometimes called Rsine, while the *śara* corresponds to $r(1 - \cos\theta)$.

In present-day mathematical terminology, the above passage says the following. Let r denote the radius of the circle, let s denote the arc and let t_n denote the nth expression obtained by applying the rule cited above. The rule requires us to calculate as follows.

1. Numerator: multiply the arc s by its square s^2, this multiplication being repeated n times to obtain $s \cdot \prod_1^n s^2$.

2. Denominator: multiply the square of the radius, r^2, by $[(2\,k)^2 \;+\; 2\,k]$ ("the squares of successive even numbers increased by that number") for successive values of k, repeating this product n times to obtain $\prod_{k=1}^{n} r^2\,[(2k)^2 \;+\; 2k]$.

Thus, the nth iterate is obtained by

$$t_n \;=\; \frac{s^{2n}\;\cdot s}{(2^2\;+\;2)\;\cdot\;(4^2\;+\;4)\cdot\;\cdots\;\cdot[(2n)^2\;+\;2n]\;\cdot\;r^{2n}}. \tag{3.1}$$

The rule further says:

$$\text{jīvā} \;=\; s - t_1 \;+\; t_2 - t_3 \;+\; t_4 - t_5 \;+\; \ldots \tag{3.2}$$

$$\;=\; s \;-\; \frac{s^3}{r^2\cdot(2^2+2)} \;+\; \frac{s^5}{r^4(2^2+2)(4^2+4)} \;-\; \cdots\,. \tag{3.3}$$

Substituting
(1) jīvā $\equiv\; r\,\sin\,\theta$,
(2) $s \;=\; r\,\theta$, so that $s^{2n\,+\,1}\,/\,r^{2n} \;=\; r\,\theta^{2n+1}$, and noticing that
(3) $[(2k)^2+2k] \;=\; 2k\cdot(2k+1)$, so that
(4) $(2^2+2)(4^2+4)\cdots[(2n)^2\;+\;2n] \;=\; (2n\;+1)!$,
and cancelling r from both sides, we see that this is entirely equivalent to the well-known expression

$$\sin\,\theta \;=\; \theta \;-\; \frac{\theta^3}{3!} \;+\; \frac{\theta^5}{5!} \;-\; \frac{\theta^7}{7!} \;+\; \cdots\,. \tag{3.4}$$

A similar rule gives an iterative expression for *śara*. The passage[14] reads:

निहत्य चापवर्गेण रूपं तत्तत्फलानि च ।
हरेद् विमूल्युग्वर्गैस्त्रिज्यावर्गहतैः क्रमात् ॥ ४४२॥
किन्तु व्यासदलेनैव द्विघ्नेनाद्यं विभज्यताम ।
फलान्यधोऽधः क्रमशो न्यस्योपर्युपरि त्यजेत् ॥ ४४३॥
शरापत्यै, संग्रहोस्यैव स्तेन स्त्रीत्यादिना क्रितः

This may be translated as follows.

> Multiply the square of the arc by the unit (= radius), and take the result of repeating that [any number of times]. Divide [each of the above numerators] by the squares of successive even numbers decreased by that number and multiplied by the square of the radius. But, the first term is [now] [the one which is] divided by twice the radius. Place the successive results so obtained one below the other and subtract [lit. remove] each from the one above. These together give the *śara*, as collected together in the verse beginning "*stena*", "*strī*", etc.

This amounts to

$$u_n = \frac{s^{2n} \cdot r}{(2^2-2)\cdot(4^2-4)\cdot \cdots \cdot [(2n)^2-2n]\,\cdot\, r^{2n}}, \tag{3.5}$$

and the rule further says that

$$\text{śara} = r(1-\cos\theta) = u_1 - u_2 + u_3 - u_4 + u_5 - \cdots, \tag{3.6}$$

$$= \frac{s^2\cdot r}{r^2(2^2-2)} - \frac{s^4 \cdot r}{r^4(2^2-2)(4^2-4)} + \cdots. \tag{3.7}$$

Recalling that
(1) śara = $r(1-\cos\theta)$,
(2) $s = r\,\theta$, so that $s^{2n}r\,/\,r^{2n} = r\,\theta^{2n}$, and noticing that
(3) $[(2k)^2-2k] = 2k\cdot(2k-1)$, so that
(4) $(2^2-2)(4^2-4)\cdots[(2n)^2 - 2n] = (2n)\,!$,
we see that this is again equivalent to the well-known expression

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots. \tag{3.8}$$

Though a great achievement in itself, the actual numerical calculation of the sine and cosine values from here is far from trivial. If too few terms are taken, the results are inaccurate, especially for larger values of the arc. If too many terms are taken, the computations become impossibly unwieldy. Even with an ordinary calculator, calculating 20! is difficult, and an accurate value of 200! is non-trivial even on a computer. Further, as we shall see later, the value of the radius r is inextricably tied to the value of π, since the circle was traditionally taken as having a fixed circumference of $21,600'$ ($= 3600\times 60$). Finally, for $\theta > 1$ (radian), i.e., for angles larger than about 58°, the value of the powers of $s\,/\,r$ goes on increasing instead of decreasing. Therefore, a numerically efficient method was evolved, which could be used to calculate the desired sine values to high accuracy with 1 division, 6 multiplications, and 5 subtractions, or just 12 arithmetical operations in all, even by those who did not use the precise value of the radius.

This required a transformation of the above series. The series (3.3) was rewritten as

$$\text{jīvā} = s - \frac{s^3}{r^3}\cdot\frac{r}{(2^2+2)} + \frac{s^5}{r^5}\cdot\frac{r}{(2^2+2)(4^2+4)} + \cdots \tag{3.9}$$

$$= s - \frac{s^3}{c^3}\cdot\frac{r\left(\frac{\pi}{2}\right)^3}{(2^2+2)} + \frac{s^5}{c^5}\cdot\frac{r\left(\frac{\pi}{2}\right)^5}{(2^2+2)(4^2+4)} + \cdots, \tag{3.10}$$

where $c = 5400'$ was a quarter of the circumference of the standard circle.

Thus, the actual calculation of sine values used a "ready-reckoner" stored "table" of numerical coefficients encapsulated in a single verse[15] of four lines beginning with *vidvān* etc.