

विद्वांसु तुन्बलः कवीशनिचयः सर्वार्थशीलस्थिरो
 निर्विद्धाङ्गनरेन्द्ररुड् निगदितेष्वेषु क्रमात् पञ्चसु ।
 आधस्त्याद् गुणितादभीष्टधनुषः कित्या विहृत्यान्तिम-
 स्याप्तं शोध्यमुपर्युपर्यथ घनेनैवं धनुष्यन्ततः ॥ ४३७ ॥

Here *vidvān*, *tunnabala*, *kaviśanicaya*, *sarvārthaśīlasthira*, and *nirviddhāṅganarendrarū* are expressions for five numbers in the reverse *kaṭapayādi*-sexagesimal system (box 3.1).

Box 3.1. *Kaṭapayādi* system

This system is based on the letters of the Sanskrit alphabet. It was known to Āryabhaṭa who had a different system. The consonants (alphabets) in due succession denote the numerals, as in the following table.

1	क	ट	प	य	1	ka	ṭa	pa	ya
2	ख	ठ	फ	र	2	kha	ṭha	pha	ra
3	ग	ड	ब	ल	3	ga	ḍa	ba	la
4	घ	ढ	भ	व	4	gha	ḍha	bha	va
5	ङ	ण	म	श	5	ṅa	ṇa	ma	śa
6	च	त		ष	6	ca	ta		ṣa
7	छ	थ		स	7	cha	tha		sa
8	ज	द		ह	8	ja	da		ha
9	झ	ध		ळ	9	jha	dha		ḷa
0	ञ	न			0	ñā	nā		

- The vowels *standing by themselves* also denote 0.
- Of two conjoint consonants, only the last has numerical significance.
- The numerals may be in direct or reverse order. (The reverse order apparently found greater favour, according to the maxim: *anīkānām vāmato gati*.)
- E.g. *bhavatī* = *bha va tī* = 4 4 6 = 644 in reverse *kaṭapayādi*.
- Chronograms may occasionally have an ordinary meaning. This additional meaning is regarded as an ornament (*śleṣa alaṅkāra*) to verse, and helps to memorize it.

In sexagesimal notation, a number is to be interpreted in terms of first (*kalā*), second (*vikalā*), and third (*tatparā*) minutes. Thus, *vidvān* = *vi dvā n* = 4 4 0 = 0''44''' (for *dvā* use the conjoint consonant rule), *tunnabala* = *tu nna ba la* = 6 0 3 3 = 33''06''', *kaviśanicaya* = *ka vī śa ni ca ya* = 1 4 5 0 6 1 = 16'05''41''', *sarvārthaśīlasthira* = *sa rva rth śī la sthi ra* = 7 4 7 5 3 7 2 = 273'57''47''', *nirviddhāṅganarendrarū* = *ni rvi ddha ṅga na re ndra ru* = 0 4 9 3 0 2 2 2 = 2220'39''40'''.

The passage may now be translated.¹⁶ The values of the Rsine, as collected together in the *śloka* beginning *vidvān* etc., are given by

vidvān, tunnabala... Successively multiply these five numbers in order by the square of the arc divided by the quarter of the circumference [i.e., 5400'], and subtract from the next number. [Continue this process with the result so obtained and the next number.] Multiply [the final result] by the cube of the arc divided by quarter of the circumference, and subtract from the arc.

In present-day notation, if we denote these numbers, starting from *vidvān* respectively by a_{11} , a_9 , a_7 , a_5 , a_3 , then if s ($\equiv r\theta$) is the given arc in minutes, c is the length ($= 5400'$) of a quadrant of the standard circle, and $jyā = r \cdot \sin \theta$, then the verse corresponds to an iterative procedure. Starting with *vidvān* (a_{11}), multiply it by $(s/c)^2$ and subtract it from the next number: *tunnabala* (a_9). Again multiply the result by $(s/c)^2$ and subtract from the next number. Multiply the final result by $(s/c)^3$ and subtract from the arc s , to obtain the *jyā*. Thus, the result may be expressed by the formula

$$r \sin \theta = s - \left(\frac{s}{c}\right)^3 \left[a_3 - \left(\frac{s}{c}\right)^2 \left[a_5 - \left(\frac{s}{c}\right)^2 \left[a_7 - \left(\frac{s}{c}\right)^2 \left[a_9 - \left(\frac{s}{c}\right)^2 a_{11} \right] \right] \right] \right]. \quad (3.11)$$

This formula is a *numerically efficient* way to approximate the sine function by its “Taylor” polynomial of the 11th order.

There is a similar formula for cosine in the next verse¹⁷ beginning with *stena* etc., corresponding to numerically efficient approximation by its “Taylor” polynomial of the 12th order.

स्तेनः स्त्रीपिशुनः सुगन्धिनगनुद् भद्राङ्गभव्यासनो
मीनाङ्गोनरसिंह ऊनधनकृद्भूरेव षटस्वेषु तु ।
आधस्त्याद् गुणितादभीष्टधनुषः कृत्या विहृत्यान्तिम-
स्याप्तं शोध्यमुपर्युपर्यथ फलं स्यादुत्क्रमस्यान्त्यजम् ॥ ४३८ ॥

This may be translated:

The six: *stena* [60 = 06'''], *strīpiśuna* [2150 = 05''12'''], *sugandhinaganud* [739030 = 03'09''37'''], *bhadrāṅgabhavyāsana* [4234170 = 071'43''24'''], *mināṅgonarasimha* [5030278 = 872'03''05'''], *ūnadhanakṛdbhūreva* [00901424 = 4241'09''00''']. Multiply by the square of the arc divided by the quarter of the circumference, and subtract from the next number. [Continue with the result and the next number.] The final result will be the *utkramajyā* [R versed sine].

This corresponds to the formula

$$r(1 - \cos \theta) = \left(\frac{s}{c}\right)^2 \left[a_2 - \left(\frac{s}{c}\right)^2 \left[a_4 - \left(\frac{s}{c}\right)^2 \left[a_6 - \left(\frac{s}{c}\right)^2 \left[a_8 - \left(\frac{s}{c}\right)^2 \left[a_{10} - \left(\frac{s}{c}\right)^2 a_{12} \right] \right] \right] \right] \right] \right] , \quad (3.12)$$

with a_k being the six numbers in the reverse *katapayādi* sexagesimal system, collected together in the verse beginning with *stena*. From (3.10), the above numbers correspond to

$$a_k = \frac{r}{k!} \left(\frac{\pi}{2}\right)^k , \quad (3.13)$$

where r is the radius of a circle of circumference 21,600 (= 3600 × 60) minutes ($r = \frac{10800}{\pi}$). The actual calculation of a_k thus requires primarily the value of π . Nīlakaṇṭha, in his *ĀryabhaṭīyaBhaṣya*,¹⁸ in his commentary on *Gaṇita* 10, described Mādhava’s subtle value of π as follows:

सङ्गमग्रामजो माधवः पुनरत्यासन्नां परिधिसंख्यामुक्तवान् –

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥

The numbers in this verse are according to a different number system, known as the *bhūta saṁkhyā* system, which uses word numerals. Thus, *netra* means 2 because one has two eyes, *veda* = 4, *guṇa* = 3, *tri* = 3, etc.¹⁹ The quantity *nikharva* = 10¹¹. Thus, the above verse may be translated:

Mādhava of Saṅgamagrāma spoke the approximate [*āsanna*] number of the circumference of a circle: *vibudha* [33] *netra* [2] *gaja* [8] *ahi* [8] *hutāśana* [3] *tri* [3] *guṇa* [3] *veda* [4] *bhavāraṇa* [27] *bāhavaḥ* [28], i.e., [2,827,433,388,233] is the measure of a circle of diameter *nava* [9] *nikharva* [100,000,000,000].

This corresponds to $\pi = 3.141, 592, 653, 5922 \dots$, accurate to 11 decimal places, with the 12th and 13th places (92 respectively) differing slightly from their accurate value (89). The term *nikharva* continues the series, *koṭi*, *arbuda*, *abja*, *kharva*, *nikharva*, then in common use for centuries.²⁰ (This decimal series coming from Vedic times is constant up to the term *koṭi* = crore, in current use. From *koṭi* onwards there are usually variations. Currently, of course, an *arbuda*, called “arab” is 100 crores, while a *kharva*, called “kharab” is 100 arabs.) The more common *katapayādi*-sexagesimal expression for r is *Devo viśvasthalī bhṛguḥ*, corresponding (in reverse order) to 34374448 or 3437' 44'' 48''' , which is still substantially more accurate than Bhāskara I’s figure of 3438', or Vaṭeśvara’s figure of 3437' 44'' .

According to Rajagopal and Rangachari, there is a significant discrepancy of $-1'''$ in the value of *mīnaṅgonarasimha*, “from its accurately rounded off [value]”.²¹ What they presumably mean is that if we use the “standard”²² rounding procedure while applying formula (3.13), then the fourths turn out to be -30 , while the fifths turn out to be -49 . Thus, the “standard” rounding procedure would round -49 fifths to -1 fourths, and when this -1 is added to -30 , we should get -31 fourths, which should then be rounded up to give another -1 thirds. Though this particular example is a bit stretched, and also involves negative numbers for which rounding conventions, even today, may vary significantly between mathematicians and computer scientists, we consider the details of the rounding procedure later on.

More to the point, the procedure used to calculate the coefficients explains the degree of accuracy actually needed for the value of π , to compute the coefficients accurately to the third minute. If it is accurate to only 7 places after the decimal, then there is an inaccuracy of $1'''$ —in the coefficient *nirviddhānanarendraru*. Thus, an accuracy of at least 8 places after the decimal point is needed for the calculation of the coefficients. In actual fact, the value of π stated is accurate to 11 decimal places.

Using the series expansion and the stored coefficients, the actual sine values are computed. These are stated in the *ĀryabhaṭīyaBhāṣya*,²³ and also in a verse in the *Laghuvivṛti* commentary on the *Tantrasaṅgraha*.²⁴

श्रेष्ठं नाम वरिष्ठानां हिमाद्रिवेदभावनः ।
 तपनो भानुसूक्तज्ञो मध्यमं विद्धि दोहनम् ॥
 धिगाज्यो नाशनं कष्टं छन्नभोगाशयाम्बिका ।
 म्रिगाहारो नरेशोऽयं वीरो रणजयोत्सुकः ॥
 मूलं विशुद्धं नाळस्य गानेषु विरळा नराः ।
 अशुद्धिगुप्ता चोरश्रीः शङ्कुकर्णो नगेश्वरः ॥
 तनुजो गर्भजो मित्रं श्रीमानत्र सुखी सखे ! ।
 शशी रात्रौ हिमाहारो वेगज्ञः पथि सिन्धुरः ॥
 छायालयो गजो नीलो निर्मलो नास्ति सत्कुले ।
 रात्रौ दर्पशमभ्राङ्गं नागस्तुङ्गनखो बली ॥
 धीरो युवा कथालोलः पूज्यो नारीजनैर्भगः ।
 कन्यागारे नागवल्ली देवो विश्वस्थली भृगुः ॥
 तत्परादिकलान्तास्तु महाज्या माधवोदिताः ।
 स्वस्वपूर्वविशुद्धे तु शिष्टास्तत्खण्डमौर्विकाः ॥ २.१.५ ॥

The numbers here are again in the reverse sexagesimal *kaṭapayādi* notation, and give the minutes, seconds, and thirds for the 24 sine values. This passage may be translated and the resulting conversion to present-day notation is given in Table 3.1.

For the sake of comparison, the numbers have also been converted to decimals, and Table 3.2 gives the comparison with sine values. A computer program was used to generate the T_EX output for this table, to avoid typing errors. It is clear that a minimum accuracy

of as high as 7 places after the decimal point is maintained for *all* values, over the entire quadrant.

Table 3.1: Mādhava's sine table.

No.	Kaṭapayādi	kalā (')	vikalā('')	tatparā('')
1	श्रेष्ठं नाम वरिष्ठानां	224	50	22
2	हिमाद्रिवेदभावनः	448	42	58
3	तपनो भानुसूक्तज्ञो	670	40	16
4	मध्यमं विद्धि दोहनम्	889	45	15
5	धिगाज्यो नाशनं कष्टं	1105	01	39
6	छन्नभोगाशयाम्बिका	1315	34	07
7	मिगाहारो नरेशोऽयं	1520	28	35
8	वीरो रणजयोत्सुकः	1718	52	24
9	मूलं विशुद्धं नाळस्य	1909	54	35
10	गानेषु विरळा नराः	2092	46	03
11	अशुद्धिगुप्ता चोरश्रीः	2266	39	50
12	शङ्कुकर्णो नगेश्वरः	2430	51	15
13	तनुजो गर्भजो मित्रं	2584	38	06
14	श्रीमानत्र सुखी सखे	2727	20	52
15	शशी रात्रौ हिमाहारो	2858	22	55
16	वेगज्ञः पथि सिन्धुरः	2977	10	34
17	छायालयो गजो नीलो	3083	13	17
18	निर्मलो नास्ति सत्कुले	3176	03	50
19	रात्रौ दर्पणमभ्राङ्गं	3255	18	22
20	नागस्तुङ्गनखो बली	3320	36	30
21	धीरो युवा कथालोलः	3371	41	29
22	पूज्यो नारीजनैर्भगः	3408	20	11
23	कन्यागारे नागवल्ली	3430	23	11
24	देवो विश्वस्थली भृगुः	3437	44	48

This raises various questions. How was the accurate value of π calculated? How were the power series obtained? Why was such a high level of accuracy required? etc. All these questions are addressed in the sequel.

Although the series expansion is clearly regarded as indefinite, the values of the coefficients, being given only to the nearest third minute, are tied to the assumption of an 11th/12th order polynomial. Hence, to achieve higher accuracy by using a higher-order polynomial, it would also be necessary to recompute the above coefficients to the desired level of accuracy. This was not done, except presumably for demonstration purposes, in the 19th c. CE *Sadratnamālā*, which computed the value of π to 17 decimal places. The limiting accuracy of the third sexagesimal minute is clearly set by the practical concerns of timekeeping, which are taken up in the next chapter, and related planetary models (not considered in detail in this book).

Table 3.2: Accuracy of Mādhava's sine table.

No.	Mādhava's sine value	Difference
1	0.0654031452	0.0000000160
2	0.1305262297	0.0000000375
3	0.1950903240	0.0000000020
4	0.2588190035	-0.0000000416
5	0.3214394797	0.0000000144
6	0.3826834083	-0.0000000241
7	0.4422886665	-0.0000000237
8	0.5000000000	0.0000000000
9	0.5555702346	0.0000000016
10	0.6087614077	-0.0000000213
11	0.6593458183	0.0000000032
12	0.7071068355	0.0000000543
13	0.7518398680	0.0000000605
14	0.7933533335	-0.0000000068
15	0.8314696287	0.0000000164
16	0.8660254521	0.0000000483
17	0.8968727739	0.0000000324
18	0.9238795632	0.0000000307
19	0.9469301920	0.0000000625
20	0.9659258390	0.0000000127
21	0.9807852980	0.0000000176
22	0.9914448967	0.0000000353
23	0.9978589819	0.0000000587
24	1.0000000000	0.0000000000

The context for the calculation of these trigonometric values within texts, such as the *Yuktibhāṣā*, is the calculation of the circumference of a circle *while avoiding the extraction of square roots*. We recollect that some 2000 years prior to the *Yuktibhāṣā*, in ca. -500 CE, the *śulba sūtra*-s had given an accurate value of $\sqrt{2}$, and that procedure inheres in the present-day term “surd” from the Latin term *surdus* (= deaf), which is a Latin translation of the Arabic mistranslation of the Sanskrit term *karaṇī* or *karṇa* (= diagonal), confused by Arabic translators with the other meaning of the word *karṇa* (= ear; hence “bad *karṇa*” = “bad ear” = “deaf”). This method of square-root extraction was used to compute the circumference of the circle, hence the value of π , starting with the octagon obtained by cutting the corners of a square. We recollect that Āryabhaṭa, who first stated a general algorithm for computing square roots, also probably used this octagon-doubling method to compute his value of the ratio today designated by π . The interesting thing about this octagon method is that it is distinct from the hexagon-doubling method widely used, from “Archimedes” to al Kāshī, to compute the circumference of a circle.²⁵ Historians have also failed to notice that